Solutions

جامعة الملك سعود
King Saud University

College of Science.
Department of Mathematics

كلية العلوم
قسم الرياضيات

Final Exam
Academic Year Choose an item.**- First Semester**

Exam Information معلومات الامتحان			
Course name	Models of Financial Economic		اسم المقرر
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Classroom No.			رقم قاعة الاختبار
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Student Information معلومات الطالب		
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ID number		الرقم الجامعي
Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions:

- Your Exam consists of 7 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
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تعليمات عامة:

- عدد صفحات الامتحان 7 صفحة. (باستثناء هذه الورقة)
- يجب إبقاء الهواتف والساعات الذكية خارج قاعة الامتحان.
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هذا الجزء خاص بأستاذ المادة
This section is ONLY for instructor

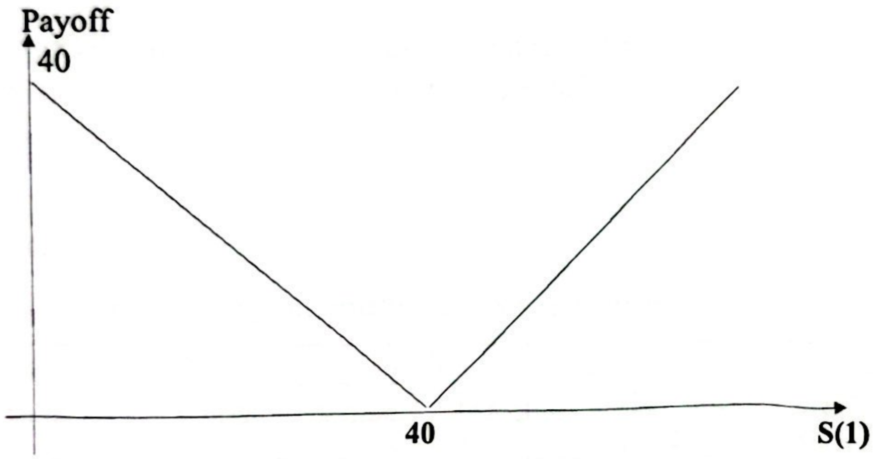
#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1	Describe some models of option pricing in financial markets			
2	Demonstrate mastery of fundament concepts of option pricing in discrete and continuous cases			
3	Apply option pricing methods to evaluate derivatives			
4	Control risk by using options in a hedging context.			
5	Model the Black Scholes option pricing for derivatives			
6	Demonstrate commitment to actuarial professional and academic values			
7	Prepare students to International Exams			
8				

EXAM COVER PAGE

Exercise 1. [6]

Assume the Black-Scholes framework. Consider a 1-year European contingent claim on a stock. You are given:

(i) The time-0 stock price is 40.
(ii) The volatility of the stock is 25%.
(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.
(iv) The continuously compounded risk-free interest rate is 6%.
(v) The time-1 payoff of the contingent claim is as follows:



1) Compute the time-0 price, delta, gamma, and theta of the contingent claim.

2) Calculate the time-0 elasticity of the contingent claim.

Exercise 2. [5]

Assume the Black-Scholes framework for a non-dividend paying stock. Let $S(t)$ denote the price at time t. You are given:

(i) $E[S(0.5)] = 120$
(ii) $Var[S(0.5)] = 700$
(iii) The expected rate of appreciation of the stock is 7%.
(iv) The continuously compounded risk-free interest rate is 4%.

Calculate the instantaneous mean return and the volatility of a 6-month 120-strike American call on the stock.

Exercise 3. [5]

Assume the Black-Scholes framework. Let $S(t)$ be the time-t price of a stock. You are given:

(i) The stock pays dividends continuously at a rate proportional to its price.
(ii) $Var[\ln S(t)] = 0.01t$, $t > 0$.
(iii) The continuously compounded risk-free rate is 7%.

Consider a contingent claim that pays $S^3(T)$ at time $T > 0$. The time-t price of the contingent claim is $V(S(t), t)$ for $0 \leq t < T$.

If $V(S(t), t) = S^3(t)\, e^{0{,}02(T-t)}$, calculate the dividend yield on the stock.

(Hint. We recall that the time-t price of a contingent claim that pays $S^a(T)$ at time T, is equal to: $S^a(t)\exp\left[(-r + a(r - \delta) + 0.5a(a-1)\sigma^2)(T - t)\right]$, where a is a constant)

Exercise 4. [4]

You are given the following information on two derivatives:

Derivative	Price	Delta	Gamma	Vega
A	1.1553	2.3107	2.3107	0.8781
B	0.3403	-0.9161	-0.1072	-0.0407

You form derivative C by taking positions on derivatives A and B. If derivative C has a zero delta and a Gamma of 0.5, calculate its vega.

Exercise 5. [5]

Assume the Black-Scholes framework. For a non-dividend paying Stock, you are given that:

(i) The expected rate of appreciation of the stock is 19%.
(ii) The volatility of the stock is 30%.
(iii) The Sharpe ratio of the stock price risk is 0.5.

You are also given the following information of a contingent claim written on the stock:

Price	10
Delta	0.5
Gamma	0.04
Theta	0.03

Find the current price of the stock.

Exercise 6. [5]

For a stock that pays dividends continuously at a rate 2%, you are given:

(i) The current stock price is 20.
(ii) The following information on a 1-year 18-strike put on the stock

Price	0.2888
Delta	-0.06577
Theta	-0.03674

(iii) The stock's volatility is 25%.
(iv) The continuously compounded risk-free interest rate is 7%.

By using delta-gamma approximation, calculate the change of the price of the put if the stock price jump to 21.50.

Exercise 7. [5]

Assume the Black-Scholes framework. You are given that:

(i) The time-0 stock price is 30.
(ii) The volatility of the stock is 18%.
(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 4%.
(iv) The continuously compounded risk-free rate is 9%.

Faisal just written 250 units of 32-strike 1-year calls and delta-hedged his position immediately. After 3 months, the stock price became 35, and the call price increase to 4.6345.

Calculate the three-month profit for Faisal, assuming that he can borrow or lend at the risk-free rate of interest and that he invests or pays the stock dividends by purchasing or shorting extra shares.

Exercise 8. [5]

You are given the following information on two derivatives:

Derivative	Delta	Gamma	Vega
A	0.5825	0.0651	0.0781
B	0.7773	0.0746	0.0596

Suppose you just bought 1000 units of Derivative A.

a) Determine the number of units of Derivative B and stock you should buy or sell in order to both delta-hedge and vega-hedge your position in Derivative A.
b) Determine the gamma of the hedged portfolio

Unless otherwise stated in the examination question, assume:

- The market is frictionless. There are no taxes, transaction costs, bid/ask spreads or restrictions on short sales. All securities are perfectly divisible. Trading does not affect prices. Information is available to all investors simultaneously. Every investor acts rationally and there are no arbitrage opportunities.
- The risk-free interest rate is constant.
- The notation is the same as used in *Derivatives Markets*, by Robert L. McDonald.

When using the standard normal distribution table, do not interpolate.

- Use the nearest z-value in the table to find the probability. Example: Suppose that you are to find $\Pr(Z < 0.759)$, where Z denotes a standard normal random variable. Because the z-value in the table nearest to 0.759 is 0.76, your answer is $\Pr(Z < 0.76) = 0.7764$.
- Use the nearest probability value in the table to find the z-value. Example: Suppose that you are to find z such that $\Pr(Z < z) = 0.7$. Because the probability value in the table nearest to 0.7 is 0.6985, your answer is 0.52.

In *Derivatives Markets*, $\Pr(Z < x)$ is written as $N(x)$.

The standard normal density function is

$$f_Z(x) = N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} = \frac{e^{-x^2/2}}{\sqrt{2 \times 3.14159}} = \frac{e^{-x^2/2}}{2.50663}, \qquad -\infty < x < \infty.$$

Let Y be a lognormal random variable. Assume that $\ln(Y)$ has mean m and standard deviation v. Then, the density function of Y is

$$f_Y(x) = \frac{1}{xv\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln(x) - m}{v}\right)^2\right], \qquad x > 0.$$

The distribution function of Y is

$$F_Y(x) = N\left(\frac{\ln(x) - m}{v}\right), \qquad x > 0.$$

Also,

$$\mathrm{E}[Y^k] = \exp\left(km + \frac{1}{2}k^2v^2\right),$$

which is the same as the moment-generating function of the random variable $\ln(Y)$ evaluated at the value k.

FORMULAS FOR OPTION GREEKS:

Delta (Δ)

Call: $e^{-\delta(T-t)}N(d_1)$,

Put: $-e^{-\delta(T-t)}N(-d_1)$

Gamma (Γ)

Call and Put: $\dfrac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}}$

Theta (θ)

Call: $\delta S e^{-\delta(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2) - \dfrac{Ke^{-r(T-t)}N'(d_2)\sigma}{2\sqrt{T-t}}$,

Put: Call Theta + $rKe^{-r(T-t)} - \delta S e^{-\delta(T-t)}$

Vega

Call and Put: $Se^{-\delta(T-t)}N'(d_1)\sqrt{T-t}$

Rho (ρ)

Call: $(T-t)Ke^{-r(T-t)}N(d_2)$,

Put: $-(T-t)Ke^{-r(T-t)}N(-d_2)$

Psi (ψ)

Call: $-(T-t)Se^{-\delta(T-t)}N(d_1)$,

Put: $(T-t)Se^{-\delta(T-t)}N(-d_1)$

NORMAL DISTRIBUTION TABLE

Entries represent the area under the standardized normal distribution from -∞ to z, Pr(Z<z)
The value of z to the first decimal is given in the left column. The second decimal place is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Values of z for selected values of Pr(Z<z)

z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
Pr(Z<z)	0.800	0.850	0.900	0.950	0.975	0.990	0.995

Solutions Final Exam ACTU 473

Sem 461

Ex 1 The graph correspond to the pay-off of straddle: $|S(1)-40|$

or straddle = Long 40-call + Long 40-Put

a) $d_1 = \dfrac{\ln\left(\frac{40}{40}\right) + (0.06 - 0.02 + 0.25^2 \times 0.5)\times 1}{0.25} = 0.285$

$\Rightarrow N(d_1) = 0.61218$

$d_2 = d_1 - \sigma\sqrt{T} = 0.035 \Rightarrow N(d_2) = 0.51396$

(1) The Call price is: $40e^{-0.02} \times 0.61218 - 40e^{-0.06} \times 0.51396 = \underline{4.64415}$

By Put-Call parity, the put price is: $\underline{3.10378}$

$\Rightarrow$ Price (straddle) = $\underline{7.74793}$

$\Delta(\text{call}) = N(d_1)e^{-\delta T} = 0.60006$

$\Delta(\text{put}) = -N(-d_1)e^{-\delta T} = -(1 - 0.61218)e^{-0.02} = 0.38014$

$\Rightarrow$ (1) $\Delta(\text{straddle}) = \underline{0.21992}$

$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-0.285^2 \times 0.5} = 0.38306$

$\Rightarrow \Gamma(\text{Call}) = \Gamma(\text{Put}) = \dfrac{e^{-0.02} \times 0.38306}{40 \times 0.25} = 0.03755$

$\Rightarrow$ (1) $\Gamma(\text{straddle}) = \underline{0.0751}$

By BS-Equation (1) $\theta(\text{straddle}) = rV - (r-\delta)S\Delta - \frac{1}{2}\sigma^2 S^2 \Gamma$

$\Rightarrow \underline{\theta = -49.886962}$

b) Elasticity (straddle) = $\dfrac{S\,\Delta(\text{straddle})}{V(\text{straddle})}$ (1)

$= \dfrac{40 \times 0.21992}{7.74793} = \underline{1.13537}$ (1)

(1)

Let σ be the volatility of the stock. Then:

$$Var[S(0.5)] = \{E[S(0.5)]\}^2 (\exp(0.5\sigma^2) - 1)$$

$$700 = 120^2 (\exp(0.5\sigma^2) - 1)$$

$$\Rightarrow \sigma = 0.30811 \quad (1)$$

Also,
$$E[S(0.5)] = S(0) e^{0.07 \times 0.5}$$

$$\Rightarrow S(0) = 120 e^{-0.035} = 115.873 \quad (1)$$

Since the stock pay no dividend, the price of the American Call is the same as the price of an otherwise identical European Call.

$$d_1 = \frac{\ln\left(\frac{115.873}{120}\right) + \left(0.04 + \frac{0.30811^2}{2}\right) \times 0.5}{0.30811\sqrt{0.5}}$$

(3)

$$d_1 = 0.0400978 \Rightarrow N(d_1) = 0.51599$$

$$d_2 = 0.0400978 - 0.30811\sqrt{0.5} = -0.177769$$

$$\Rightarrow N(d_2) = 0.42945$$

$$\Rightarrow C = 115.873 \times 0.51599 - 120 e^{-0.02} \times 0.42945$$

$$\Rightarrow C = 9.2758$$

$$\begin{cases} \Delta(C) = 0 \\ \Gamma(C) = 0.5 \end{cases} \qquad C: \begin{cases} x \text{ units of } A \\ y \text{ units of } B \end{cases}$$

We have:
$$\begin{cases} 2.3107x - 0.5161y = 0 \\ 2.3107x - 0.1072y = 0.5 \end{cases} \quad (3)$$

$\Rightarrow \quad y = 0.6181 \qquad x = 0.2451$

(1) $\nu(C) = 0.2451 \times 0.8781 + 0.618 \times (-0.0407)$

$\nu(C) = 0.19$

Ex5 By BS-Equation

$$rV = \Theta + (r-\delta)S\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma \quad (1)$$

Since $\phi = \dfrac{\alpha - r}{\sigma} = 0.5$ and $\alpha = 0.19$, $\sigma = 0.3$

We have: $r = 0.04$ (1)

(2)
$\Rightarrow 0.04 \times 10 = 0.03 + (0.04 - 0)S \times 0.5 + \frac{1}{2}(0.3)^2(0.04)S^2$

$\Rightarrow S = 9.82039$ or $S = -20.9315$ (rejected)

Ex3 $\sigma = 0.1 \qquad r = 0.07$

$$V(S(t), t) = S^3(t)\, e^{0.02(T-t)}$$
$$= S^{(a)}(t)\, e^{[-r + a(r-\delta) + 0.5a(a-1)\sigma^2](T-t)}$$

(3) For $a = 3$, by comparing the coefficients:

$$-r + a(r-\delta) + 0.5a(a-1)\sigma^2 = 0.02 \quad (2)$$

$$\Rightarrow \delta = 0.14$$

(3)

Ex6

By BS-Equation, we first compute Θ :

$$\Theta + (r-\delta) S \Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV$$

$$0.07 \times 0.2888 = -0.03674 + 0.05 \times 20 \times (-0.06577) + \frac{0.25^2}{2} \times 20^2 \Gamma$$

$$\Rightarrow \quad \Gamma = 0.0098181 \quad (2)$$

The approximate change is $\Delta \times \varepsilon + \frac{1}{2}\Gamma\varepsilon^2 = -0.06557 \times 1.5 + 0.5 \times 0.0098181\,(1.5)^2$

(3) $= -0.0876$

Ex 7

(5)

Time	Derivative	Hedge Portfolio: Stock	Hedge Portfolio: Cash	Aggregate
0	$V(S(t),t) = -469.475$	$-\Delta S(t) = 3629.4783$	-3160.0033	0
0.25	$V(S(t+h),t+h) = -1158.625$	$-\Delta e^{\delta h} S(t+h) = 120.98261 \times 35 \times e^{0.04/4}$	$-3160.0033\, e^{0.09/4}$	P/L: -113.5866

Ex8

x-unit of B , y-unit of stock

a) . Delta hedge : $1000\,(0.5825) + x \times 0.7773 + y = 0$ (1)

(3) . Vega hedge : $1000\,(0.0781) + x \times 0.0596 + 0 = 0$ (2)

$$\Rightarrow \begin{cases} x = 100.476 \\ y = -660.60 \end{cases}$$

Thus, to hedge, one should buy 100.476 Derivative B and short 660.60 shares

b) (2) $\Gamma = 100.476 \times 0.0746 + 0 \;(= \Gamma(\text{stock}))$

$\Gamma = 7.4955$