Solution of final examination	Math 481	Semester II - 1445

Question 1 :

1. The function $\sin(\frac{\pi}{2}x)$ is continuous, so $U(g) = 1 + \int_0^1 \sin(\frac{\pi}{2}x)dx = 1 + \frac{2}{\pi}$ and $L(g) = \int_0^1 \sin(\frac{\pi}{2}x)dx = \frac{2}{\pi}$.
2. g is not Riemann integrable.
3. $g = \sin(\frac{\pi}{2}x)$ a.e, then g is Lebesgue integrable.

Question 2 :

1. $\dfrac{1}{e^{-x}+e^{x}} \leq e^{-x}$ and $\displaystyle\int_1^{+\infty} e^{-x}dx = \frac{1}{e}$, then the integral $\displaystyle\int_1^{+\infty} \frac{dx}{e^{-x}+e^{x}}$ is convergent.
2. $\dfrac{e^{-x}}{x} \geq \dfrac{1}{ex}$ and $\displaystyle\int_0^1 \frac{1}{ex}dx = +\infty$, then the integral $\displaystyle\int_0^1 \frac{e^{-x}}{x}dx$ is divergent.

Question 3 :

1. Since the sequence $(\dfrac{1}{n+x^2})_n$ is decreasing, the $\left|\displaystyle\sum_{k=n}^{m} \frac{(-1)^k}{k+x^2}\right| \leq \dfrac{1}{n+x^2}$, for all $n \leq m \in \mathbb{N}$ and $x \in \mathbb{R}$.
2. (a) We deduce that $\displaystyle\sup_{x\in\mathbb{R}} \left|\sum_{k=n}^{m} \frac{(-1)^k}{k+x^2}\right| \leq \frac{1}{n} \underset{n\to+\infty}{\longrightarrow} 0$, then the series $\displaystyle\sum_{n\geq 1} f_n(x)$ is uniformly convergent on $\mathbb{R}$.
   (b) As the functions f_n are continuous and the convergence is uniform of $\mathbb{R}$, then f is continuous on $\mathbb{R}$.
   (c) As the convergence is uniform of the series $\displaystyle\sum_{n\geq 1} f_n(x)$ on $\mathbb{R}$ and $\displaystyle\lim_{x\to+\infty} f_n(x) = 0$, then $\displaystyle\lim_{x\to+\infty} f(x) = 0$.

Question 4 :

Define the sequence of functions $(f_n)_n$ on $[0, \infty)$ by: $f_n(x) = \dfrac{n^2x}{1+n^2x^3}$ for all $n \in \mathbb{N}$.

1. $\lim_{n\to+\infty} f_n(0) = 0$ and $\lim_{n\to+\infty} f_n(x) = \frac{1}{x^2}$, for $x \neq 0$.

2. $|f_n(x) - f(x)| = \frac{1}{x^2(1+n^2x^3)} \leq \frac{1}{n^2}$ for $x \geq 1$, then the sequence $(f_n)_n$ is uniformly convergent on the interval $[1, \infty)$.
   The functions f_n are continuous on $[0, 1]$ but the limit is not continuous, then the sequence $(f_n)_n$ is not uniformly convergent on $[0, 1]$.

3. Course

4. The sequence $(f_n)_n$ is increasing, then $\lim_{n\to\infty} \int_0^1 \frac{n^2x}{1+n^2x^3} dx = \int_0^1 \frac{dx}{x^2} dx = +\infty$.

Question 5 :

1. A subset E of $\mathbb{R}$ is said to be measurable with respect to the Lebesgue outer measure m^* if

$$m^*(X) = m^*(X \cap E) + m^*(X \cap E^c), \quad \forall X \subset \mathbb{R}.$$

2. If $m^*(E) = 0$, then for $X \subset \mathbb{R}$, $m^*(X \cap E) = 0$ and $m^*(X \cap E^c) \leq m^*(X)$. Then $m^*(X) \geq m^*(X \cap E) + m^*(X \cap E^c)$. Moreover as m^* is an outer measure, $m^*(X) \leq m^*(X \cap E) + m^*(X \cap E^c)$. Then $m^*(X) = m^*(X \cap E) + m^*(X \cap E^c)$ and E is measurable.

3. As $m^*(\{a\}) = 0$ for all $a \in \mathbb{R}$, then for any countable set E in $\mathbb{R}$, $m^*(E) = 0$.

4. As $m^*([0, 1]) = 1$, then $[0, 1]$ is not countable.