King Saud University
Faculty of Sciences
Department of Mathematics

Final Examination	Math 481	Semester II - 1445	Time: 3H

Question 1 :

Let $f\colon [0,1] \longrightarrow \mathbb{R}$ be the function defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \notin \mathbb{Q} \cap [0,1] \end{cases}$$

Define the function $g(x) = f(x) + \sin(\frac{\pi}{2}x)$, for $x \in [0,1]$.

1. Find $U(g)$ and $L(g)$.
2. Is g Riemann integrable?
3. Is g Lebesgue integrable?

Question 2 :

Determine if the following improper integrals are convergent or divergent:

1. $\displaystyle\int_1^{+\infty} \frac{dx}{e^{-x} + e^{x}}$
2. $\displaystyle\int_0^1 \frac{e^{-x}}{x} dx.$

Question 3 :

Define the sequence of functions $(f_n)_n$ on $\mathbb{R}$ by: $f_n(x) = \dfrac{(-1)^n}{n + x^2}$, for all $n \in \mathbb{N}$.

1. Prove the inequality $\displaystyle\left|\sum_{k=n}^{m} \frac{(-1)^k}{k + x^2}\right| \leq \frac{1}{n + x^2}$, for all $n \leq m \in \mathbb{N}$ and $x \in \mathbb{R}$.
2. Deduce
   (a) the series $\displaystyle\sum_{n \geq 1} f_n(x)$ is uniformly convergent on $\mathbb{R}$,

(b) the function f is continuous on $\mathbb{R}$, where $f(x) = \sum_{n=1}^{+\infty} f_n(x)$,

(c) $\lim_{x \to +\infty} f(x) = 0$.

Question 4 :

Define the sequence of functions $(f_n)_n$ on $[0, \infty)$ by: $f_n(x) = \dfrac{n^2 x}{1 + n^2 x^3}$ for all $n \in \mathbb{N}$.

1. Prove that the sequence $(f_n)_n$ is convergent and find its limit.
2. Show that the sequence $(f_n)_n$ is uniformly convergent on the interval $[1, \infty)$ but is not uniformly convergent on $[0, 1]$.
3. State the Monotone Convergence Theorem.
4. Evaluate $\lim_{n \to \infty} \int_0^1 \dfrac{n^2 x}{1 + n^2 x^3} dx$.

Question 5 :

1. State the definition of a measurable set with respect to the Lebesgue outer measure m^*.
2. Prove that if $m^*(E) = 0$, then E is measurable.
3. Prove that $m^*(E) = 0$ for any countable set E in $\mathbb{R}$.
4. Deduce that $[0, 1]$ is not countable.