Solution of the Final Exam	Math 280	December 2025

Question 1 :

1. We can take $u_n = 1 + \frac{\sqrt{2}}{n}$.
2. We can take $v_n = 1 - \frac{\sqrt{2}}{n}$ and $w_n = 1 - \frac{\sqrt{2}}{n}$.

Question 2 :

1. $\lim_{n\to\infty} \frac{n^2}{n^2+\cos(n^2)} = 1$, then the series $\sum_{n\geq 1} \frac{1}{n^2+\cos(n^2)}$ is convergent.
2. $\lim_{n\to\infty} n\frac{2n+1}{n^2+2n+3} = 2$, then the series $\sum_{n\geq 1} \frac{2n+1}{n^2+2n+3}$ is divergent;
3. $\lim_{n\to\infty} n^{\frac{3}{2}}\frac{|\sin(n)\ln(n)|}{n^2} = 0$, then the series $\sum_{n\geq 2} \frac{\sin(n)\ln(n)}{n^2}$ is convergent.
4. $\lim_{n\to\infty} \left(\frac{3}{2}\right)^n \frac{1+2^n}{1+3^n} = 1$, then the series $\sum_{n\geq 1} \frac{1+2^n}{1+3^n}$ is convergent;

Question 3 :

1. $f'(x) = 2 - \sin x > 0$, then f is increasing on $\mathbb{R}$;
2. As f is increasing, $f([0,\pi]) = [f(0), f(\pi)] = [1, 2\pi - 1]$.

Question 4 :
$f'(x) = 3x^2 - 6x + 4$, the discriminant $\Delta = 36 - 48 < 0$, then $f'(x) > 0$ for all $x \in \mathbb{R}$. As $f(0) = -1$ and $f(1) = 1$, then there exists $c \in [0,1]$ such that $f(c) = 0$. c is unique because f is strictly increasing.

Question 5 :

1. Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function and differentiable on (a,b). If $f(a)=f(b)$, there exists $c \in (a,b)$ such that $f(c)=f'(c)$.

2. $g'(x) = e^x(f(x) - f'(x) + f'(x) - f''(x)) = e^x(f(x) - f''(x))$. As $g(a) = g(b) = 0$, then there exists $c \in (a,b)$ such that $g'(c) = 0 \iff f(c) = f'(c)$.

Question 6 :

1. $\sin x - \sinh x = -\frac{x^3}{6}(\cos c + \cosh c)$.
   $\sin x.\sinh x = x^2 + \frac{x^3}{3}(-\sin c \cosh c + \cos c \sinh c)$.

2.

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{\sinh x}\right) &= \lim_{x\to 0}\left(\frac{\sin x - \sinh x}{\sin x.\sinh x}\right) \\
&= \lim_{x\to 0}\left(\frac{-\frac{x^3}{6}(\cos c + \cosh c)}{x^2 + \frac{x^3}{3}(-\sin c' \cosh c' + \cos c' \sinh c')}\right) = 0.
\end{aligned}$$