King Saud University
Faculty of Sciences
Department of Mathematics

Final Examination	Math 244	Semester II - 1445	Time: 3H

Name: ID: Section Signature

Note: • Attempt all the five questions. Scientific calculators are not allowed.
• Please fill in the above columns and give your answer to Question 1 on the following table.

Question	*i)*	*ii)*	*iii)*	*iv)*	*v)*	*vi)*	*vii)*	*viii)*	*ix)*	*x)*	Marks
Choice											

Question 1 : [Marks: 10×1]: Choose the correct answer:

i) If B is a 5×7 matrix and nulity$(B) = 3$, then nulity(B^T) equals:

a) 2 b) 5 c) 3 d) 1

ii) If C is a 3×3 matrix with $\det(C) = 2$, then rank(C) equals:

a) 1 b) 3 c) 2 d) 0

iii) Which of the following matrices cannot be a transition matrix:

a) $\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & -3 \\ 0 & 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 0 & 0 \\ -2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 5 \\ 2 & 6 & 0 \end{pmatrix}$.

iv) If u and v are orthogonal unit vectors in an inner product space, then$\|v - u\|$ equals:

a) 1 b) $\sqrt{2}$ c) 0 d)$\sqrt{3}$

v) If$u = (1, -1, 0), v = (3, -1, 2)$ and $w = (-2, 0, 1)$ in the Euclidean space $\mathbb{R}^3$, then $\frac{\langle u, v\rangle}{\langle w, w\rangle} v$ equals:

a) $\frac{4}{5}(3, -1, 2)$ b) $\frac{4}{5}$ c) $\frac{5}{4}(-2, 0, 1)$ d) $\frac{4}{5}(1, -1, 0)$

vi) If v and w are orthogonal nonzero vectors in the Euclidean space $\mathbb{R}^n$, then the angle θ between v and w equals:

a) 45^o b) 90^o c) 180^o d) 0^o

vii) The set $\{(x, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)\}$ of vectors in the Euclidean space $\mathbb{R}^3$ is (ortho)normal if x equals:

a) 0 b) $\frac{1}{\sqrt{6}}$ c)$\frac{1}{\sqrt{6}}$ d) $\frac{1}{2}$.

viii) Consider the vector space $\mathscr{P}_n$ of real polynomials of degree $\leq n$. I f the linear transformation $T\colon \mathscr{P}_2 \longrightarrow \mathscr{P}_1$ is given by $T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1 + a_2 x$, then $\ker(T)$ equals:

a) $\{0\}$ b) $\text{span}\{-1 + x\}$ c) $\text{span}\{-1, x\}$ d) $\mathscr{P}_2$

ix) Consider the basis $\{(1,2),(3,0)\}$ for the space $\mathbb{R}^2$. If the linear transformation $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $T(1,2) = (1,5)$ and $T(3,0) = (-4,6)$, then $T(7,8)$ equals:

a) (13,26) b) (-15,35) c) (-3,11) d) (0,26)

x) If $A = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}$, then the eigenvalues of A^4 are:

a) 2,16 b) -1,8 c) 1,16 d) 4,16.

Question 2 : [Marks: 2+2+3]

a) Compute $P(A)$, if P is the polynomial $P(x) = x^3 - 3x^2 - 4x + 13$ and $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

b) Show that $\begin{vmatrix} 1 & 2 & 1 \\ x+1 & 2x+1 & 2x+2 \\ x+1 & x+1 & 2x+1 \end{vmatrix}$ is constant, for all $x \in \mathbb{R}$.

c) Let A be $m \times n$ matrix and let B and C be linearly independent vectors in $\mathbb{R}^n$. Suppose X_1 is a solution of $AX = B$ and X_2 is a solution of $AX = C$. Then show that X_1 and X_2 are linearly independent in $\mathbb{R}^n$.
Explain why $\text{rank}(A) \geq 2$.

Question 3 : [Marks: 2+4+2]

Consider the matrix $A = \begin{pmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{pmatrix}$.

a) Compute the RREF of A.

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b) Find a basis B of the column space Col(A) of the matrix A and a basis C of the null space $N(A)$ of the matrix A.

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c) Find rank(A) and Nullity(A).

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Question 4 : [Marks: 3+2+3]

a) Consider the vector space $\mathscr{P}_2$ of real polynomials of degree ≤ 2 equipped with the inner product:

$$\langle P(x), Q(x)\rangle = P(-1)Q(-1) + P(0)Q(0) + P(1)Q(1), \quad \forall P, Q \in \mathscr{P}_2.$$

Explain why $\{1, x, x^2\}$ is not orthogonal. Apply the Gram-Schmidt process to transform the basis $\{1, x, x^2\}$ of $\mathscr{P}_2$ to an orthogonal basis.

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b) Let $S = \{u, v, w\}$ be any orthonormal subset of the above inner product space $\mathscr{P}_2$. Show that S is a basis for $\mathscr{P}_2$.

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c) Consider the basis $\{v_1 = (2, 2, 1), v_2 = (2, 1, 0), v_3 = (1, 0, 0)\}$ for the vector space $\mathbb{R}^3$. Let the linear transformation $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by:$T(v_1) = (3, -1)$, $T(v_2) = (6, 2)$ and $T(v_3) = (4, 3)$. Then find rank(T) and nullity(T).

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Question 5 : [Marks: 3+4]

a) Let A, B and C be square matrices of size n, where C is invertible satisfying $B = C^{-1}AC$. If λ is an eigenvalue of A and X is its corresponding eigenvector, then find a nonzero vector $Y \in \mathbb{R}^3$ such that $BY = \lambda Y$.

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b) Show that the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}$ is diagonalizable and find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.