Solution of the Final Examination	Math 244	Semester II - 1445

Question 1 :

Question	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)
Solution	d)	b)	d)	b)	a)	b)	b)	b)	d)	c)

Question 2 : [Marks: 2+2+3]

a) $P(A) = I_3$.

b) $\begin{vmatrix} 1 & 2 & 2 \\ x+1 & 2x+1 & 2x+2 \\ x+1 & x+1 & 2x+1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x+1 & -1 & 1 \\ x+1 & -x-1 & x \end{vmatrix} = 1.$

c) If $aX_1 + bX_2 = 0$, then $A(aX_1 + bX_2) = aB + bC = 0$, then $a = b = 0$, since B, C are linearly independent.
B and C are in the image space of A and linearly independent. Then $\text{rank}(A) \geq 2$.

Question 3 : [Marks: 2+4+2]

a) The RREF of A is the matrix $\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

b) $B = \{(1, -1, 0, 1), (-1, 1, 1, 0)\}$ i a basis of the column space $\text{Col}(A)$ and $C = \{(1, 1, -1, 0), (-2, -3, 0, 1)\}$ is a basis of the null space $N(A)$.

c) $\text{rank}(A) = 2$ and $\text{Nullity}(A) = 2$.

Question 4 : [Marks: 3+2+3]

a) $\langle 1, x^2 \rangle = 2$. Then $\{1, x, x^2\}$ is not orthogonal.
$(\frac{1}{\sqrt{3}}, \frac{x}{\sqrt{2}}, \frac{1}{\sqrt{6}}(3x^2 - 2))$ is the orthonormal basis.

b) If $au + bv + cw = 0$, then taking the inner product with u, v and w respectively, we get $a = 0$, $b = 0$ and $c = 0$.

c) The matrix of T with respect to the basis $B = \{v_1 = (2, 2, 1), v_2 = (2, 1, 0), v_3 = (1, 0, 0)\}$ and the standard basis of $\mathbb{R}^2$ is $A = \begin{pmatrix} 3 & 6 & 4 \\ -1 & 2 & 3 \end{pmatrix}$. Then $\text{rank}(T) = 2$ and $\text{nullity}(T) = 1$.
The set of solutions of the system $AX = 0$ is $\{(10, -13, 12)t : \ t \in \mathbb{R}\}$. Then

$$\ker(T) = \{(10v_1 - 13v_2 + 12v_3)t : \ t \in \mathbb{R}\}.$$

Question 5 : [Marks: 3+4]

a) $Y = C^{-1}X$.

b)

$$\begin{aligned} q_A(\lambda) &= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 2-\lambda & 2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (2+\lambda)\begin{vmatrix} -1 & 2 & 3 \\ 0 & 2-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} \\ &= (2+\lambda)\begin{vmatrix} -1 & 2 & 3 \\ 0 & 2-\lambda & 2 \\ 0 & 4 & 4-\lambda \end{vmatrix} = -(2+\lambda)\begin{vmatrix} -\lambda & 2 \\ \lambda & 4-\lambda \end{vmatrix} \\ &= -\lambda(2+\lambda)(\lambda-6). \end{aligned}$$

For $\lambda = 0,\ X_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. For $\lambda = -2,\ X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. For $\lambda = 6,\ X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$P = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$.