(1)

Exam Answers

) Use Gomory's algorithm to solve the following Integer programming problem

" construct only the first Gomory's constraint and its tables

Max $Z = 4x_1 + 6x_2 + 2x_3$,

s.t. $4x_1 - 4x_2 \le 5$,

$-x_1 + 6x_2 \le 5$,

$-x_1 + x_2 + x_3 \le 5$

$x_1, x_2, x_3 \ge 0$

x_1 and x_3 are integers

when the optimal non-integer solution is given by the following table:

C_B	BV	x_1	x_2	x_3	s_1	s_2	s_3	b	
4	x_1	1	0	0	3/10	1/5	0	5/2	
6	x_2	0	1	0	1/20	1/5	0	5/4	
2	x_3	0	0	1	1/4	0	1	25/4	
z_j		4	6	2	2	2	2	$Z_{max} = 30$	
$\bar{c}_j$		0	0	0	-2	-2	-2		

where s_1, s_2 and s_3 in the opposite table are called slack variables.

Solution: From the table the non-integer optimal solution is

$$x_1 = 5/2 \qquad x_2 = 5/4 \quad \text{and} \quad x_3 = 25/4$$

$$\Rightarrow \quad x_1 = 2 + 1/2 \qquad x_2 = 1 + 1/4 \qquad x_3 = 6 + 1/4$$

So the largest fraction is 1/2 and hence Gomory's constraint is constructed using x_1-row and is taken the following form

$$1x_1 + \tfrac{3}{10}s_1 + \tfrac{1}{5}s_2 = 5/2$$

$$(1+0)x_1 + \left(0 + \tfrac{3}{10}\right)s_1 + \left(0 + \tfrac{1}{5}\right)s_2 = 2 + 1/2$$

The new variable $\Rightarrow \quad s' = \frac{3}{10}s_1 + \frac{1}{5}s_2 - 1/2 \quad \Rightarrow \quad -\frac{3}{10}s_1 - \frac{1}{5}s_2 + s' = -1/2$

	c_j	4	6	2	0	0	0	0		
C_B	BV	x_1	x_2	x_3	s_1	s_2	s_3	s'	b	Ratio
4	x_1	1	0	0	3/10	1/5	0	0	5/2	
6	x_2	0	1	0	1/20	1/5	0	0	5/4	
2	x_3	0	0	1	1/4	0	1	0	25/4	
0	s'	0	0	0	-3/10	-1/5	0	1	-1/2	← infeasible
z_j		4	6	2	2	2	2	0	$Z_{max} = 30$	
$\bar{c}_j$		0	0	0	-2	-2	-2	0		
CB	BV	x_1	x_2	x_3	s_1	s_2	s_3	s'	b	
4	x_1	1	0	0	0	0	0	1	2	
6	x_2	0	1	0	0	1/6	0	1/6	7/6	
2	x_3	0	0	1	0	-1/6	1	5/6	35/6	
0	s_1	0	0	0	1	2/3	0	-10/3	5/3	
z_j		4	6	2	0	2/3	2	29/3	$Z_{max} = 80/3$	

∴

$x_1 = 2$ integer

$x_2 = 7/6$ non-integer

$x_3 = 35/6$ non-integer

$Z_{max} = 80/3$

1. Solve the following mixed-integer problem by Branch-and-bound (BB) technique. Suppose the continuous solution is given by the table below.

" Solve the subproblem with $\le$ constraints only "

Max $Z = x_1 + x_2$;

s.t. $2x_1 + 5x_2 \le 16$;

$6x_1 + 5x_2 \le 30$;

$x_2 \ge 0$,

$x_1 \ge 0$ and integer

	c_j	1	1	0	0	
CB	BV	x_1	x_2	s_1	s_2	b
1	x_2	0	1	3/10	$-1/10$	9/5
1	x_1	1	0	$-1/4$	1/4	7/2
z_j		1	1	1/20	3/20	$Z_{max} = \frac{53}{10}$
$\bar{c}_j$		0	0	$-1/20$	$-3/20$	

Solution: The continuous optimum solution is: $x_1 = 7/2$, $x_2 = 9/5$

we have to focus on x_1 only (integer)

$\Rightarrow \quad x_1 = 7/2 = 3\frac{1}{2} \quad \Rightarrow \quad 3 < x_1 < 4$ then we have two subproblems with $x_1 \le 3$ and $x_1 \ge 4$

Subproblem (1)

Max $Z = x_1 + x_2$,

s.t. $2x_1 + 5x_2 \le 16$,

$6x_1 + 5x_2 \le 30$,

$x_2 \ge 0$,

$x_1 \le 3$

x_1 integer

	c_j	1	1	0	0	0		
CB	BV	x_1	x_2	s_1	s_2	s_3	b	Ratio
0	s_1	2	5	1	0	0	16	16/2
0	s_2	6	5	0	1	0	30	30/6
0	s_3	1	0	0	0	1	3	3/1 ← min
z_j		0	0	0	0	0		
$\bar{c}_j$		1	1	0	0	0	not all $\bar{c}_j \le 0$	
CB	BV	x_1	x_2	s_1	s_2	s_3	b	Ratio
0	s_1	0	5	1	0	-2	10	10/5 ← min
0	s_2	0	5	0	1	-6	12	12/5
1	x_1	1	0	0	0	1	3	none
z_j		1	0	0	0	1	$Z_{max} = 3$	
$\bar{c}_j$		0	1	0	0	-1	not all $\bar{c}_j \le 0$	
CB	BV	x_1	x_2	s_1	s_2	s_3	b	Ratio
1	x_2	0	1	1/5	0	$-2/5$	2	
0	s_2	0	0	-1	1	-4	2	
1	x_1	1	0	0	0	1	3	
z_j		1	1	1/5	0	3/5	$Z_{max} = 5$	

Subproblem (2)

Max $Z = x_1 + x_2$,

s.t. $2x_1 + 5x_2 \le 16$,

$6x_1 + 5x_2 \le 30$,

$x_1 \ge 4$

$x_2 \ge 0$

x_1 integer.

" This case is not required in the question "

it can be solved by the Big-M or the two phase method.

$\Rightarrow$ The optimal integer solution is $x_1 = 3$, $x_2 = 2$ and $Z_{max} = 5$

• the additive algorithm (Balas's algorithm) to solve the following 0-1 problem:

$$\text{Max } z = 3x_1 + x_2 + 3x_3,$$

s.t.

$$-x_1 + 2x_2 + x_3 \le 4,$$
$$4x_2 - 3x_3 \le 2,$$
$$x_1 - 3x_2 + 2x_3 \le 3,$$
$$x_1, x_2, x_3 = 0 \text{ or } 1$$

Solution: Since Max z = Min z' where $z' = -z$

Then

$$\text{Min } z' = -3x_1 - x_2 - 3x_3$$

let

$$x_1 = 1 - y_1, \quad x_2 = 1 - y_2, \quad x_3 = 1 - y_3$$

$$\Rightarrow \quad \text{Min } z' = 3y_1 + y_2 + 3y_3 - 10$$

let

$$z'' = z' + 10 = 3y_1 + y_2 + 3y_3$$

Then the modified problem is

$$\text{Min } z'' = 3y_1 + y_2 + 3y_3$$

s.t.

$$y_1 - 2y_2 - y_3 \le 2, \qquad y_1 - 2y_2 - y_3 + S_1 = 2$$
$$-4y_2 + 3y_3 \le 1, \quad \Rightarrow \quad -4y_2 + 3y_3 + S_2 = 1$$
$$-y_1 + 3y_2 - 2y_3 \le 3 \qquad -y_1 + 3y_2 - 2y_3 + S_3 = 3$$
$$y_1, y_2, y_3 = 0 \text{ or } 1$$

So we have the following:

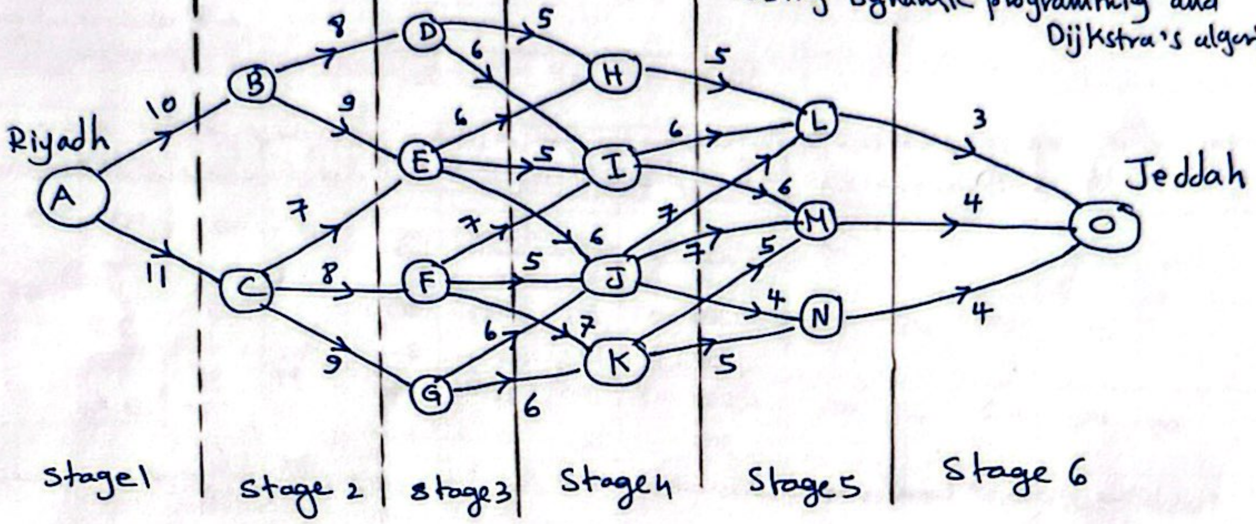
$$S_1 = 2 - y_1 + 2y_2 + y_3$$
$$S_2 = 1 + 4y_2 - 3y_3$$
$$S_3 = 3 + y_1 - 3y_2 + 2y_3$$

At the intial solution $y_1 = y_2 = y_3 = 0 \Rightarrow S_i \ge 0 \;\; \forall \; i = 1,2,3$

This means that $(y_1, y_2, y_3) = (0,0,0)$ is the best feasible solution

$$\Rightarrow \quad (x_1, x_2, x_3) = (1,1,1) \Rightarrow z_{max} = 7$$

A Salesman is planning a business tour from Riyadh to Jeddah in the course of which he proposes to cover one city from each of the Company's different market zones on route. As he has limited time at his disposal, he has to complete his tour in the shortest possible time. The network given below shows the number of day's time involved for covering any of the various intermediate cities (time includes travel as well as working time). Determine the optimum tour plan. using Dynamic programming and Dijkstra's algorithm



stage 1

$d_1 \rightarrow$	$f_1(s_1,d_1) = D_{s_1,d_1}$ A	$f_1^*(s_1)$	d_1^*
s_1 B	10	10	A
C	11	11	A

stage 2

$d_2 \rightarrow$	$f_2(s_2,d_2) = D_{s_2,d_2} + f_1^*(s_1)$ B	C	$f_2^*(s_2)$	d_2^*
s_2 D	8+10	—	18	B
E	9+10	7+11	18	C
F	—	8+11	19	C
G	—	9+11	20	C

stage 3

$d_3 \rightarrow$	$f_3(s_3,d_3) = D_{s_3,d_3} + f_2^*(s_2)$ D	E	F	G	$f_3^*(s_3)$	d_3^*
s_3 H	5+18	6+18	—	—	23	D
I	6+18	5+18	7+19	—	23	E
J	—	6+18	5+19	6+20	24	E or F
K	—	—	7+19	6+20	26	F or G

stage 4

$d_4 \rightarrow$	$f_4(s_4,d_4) = D_{s_4,d_4} + f^*(s_3)$ H	I	J	K	$f_4^*(s_4)$	d_4^*
s_4 L	5+23	6+23	7+24	—	28	H
M	—	6+23	7+24	5+26	29	I
N	—	—	4+24	5+26	28	J

stage 5

$d_5 \rightarrow$	$f_5(s_5,d_5) = D_{s_5,d_5} + f_4^*(s_4)$ L	M	N	$f_5^*(s_5)$	d_5^*
s_5 O	3+28	4+29	4+28	31	L

$L \xrightarrow{31} O$

shortest paths according to the backward method are :

stage 5 : L → O

stage 4 : H → L
J → N

stage 3 : D → H
E → I

stage 2 : B → D
C → E

stage 1 : A — B

Then the shortest tour is

A → B → D → H → L → O

with shortest time = 31 time unit.

Using Dijkstra's algorithm:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	
A	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	
B		10 AB	11 AC	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	
C			11 AC	18 ABD	19 ABE	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	
D				18 ABD	18 ACE	19. ACF	20 ACG	∞	∞	∞	∞	∞	∞	∞	∞	
E					18 ACE	19 ACF	20 ACG	23 ABDH	24 ABDI	∞	∞	∞	∞	∞	∞	
F						19 ACF	20 ACG	23 ABDH	23 ACEI	24 ACEJ	∞	∞	∞	∞	∞	
G							20 ACG	23 ABDH	23 ACEI	24 ACEJ 24 ACFJ	26 ACFK	∞	∞	∞	∞	
H								23 ABDH	23 ACEI	24 ACEJ 24 ACFJ	26 ACFK 26 ACGK	∞	∞	∞	∞	
I									23 ACEI	24 ACEJ 24 ACFJ	26 ACFK 26 ACGK	28 ABDHL	∞	∞	∞	
J										24 ACEJ ACFJ	26 ACFK ACGK	28 ABDHL	29 ACEIM	∞	∞	
K											26 ACFK ACGK	28 ABDHL	29 ACEIM	28 ACEJN ACFJN	∞	
L												28 ABDHL	29 ACEIM	28 ACEJN ACFJN	∞	
M													29 ACEIM	28 ACEJN, ACFJN	31 ABDHLO	
N													29 ACEIM		31 ABDHLO	
O															31 ABDHLO	

on the shortest tour according to Dijkstra's algorithm is
$A \to B \to D \to H \to L \to O$ with shortest time $= 31$ unit

(5) Use the dynamic programming to solve the following problem:

$$\text{Max} \prod_{i=1}^{3} u_i$$

$$\text{s.t.} \quad \sum_{i=1}^{3} u_i = 10, \quad u_i \geq 0 \; ; \; i = 1,2,3$$

Solution: This is a three-stage problem $\Rightarrow$ Suppose x_1, x_2 and x_3 be the state variables then.

stage 3:

Let $x_3 = \sum_{i=1}^{3} u_i = u_1 + u_2 + u_3 = 10$

stage 2: Let $x_2 = u_1 + u_2 \Rightarrow x_2 = x_3 - u_3$

Stage 1: Let $x_1 = u_1 \Rightarrow x_1 = x_2 - u_2$

$$\Rightarrow f_1(x_1) = \text{Max } u_1 \Rightarrow f(x_1) = u_1 = x_2 - u_2$$

from stage 2 $\Rightarrow$ $f_2(x_2) = \text{Max } u_1 u_2 = \underset{0 \leq u_2 \leq x_2}{\text{Max}} \{u_2 (x_2 - u_2)\}$

Let $\phi = u_2(x_2 - u_2) = u_2 x_2 - u_2^2$

$$\frac{\partial \phi}{\partial u_2} = x_2 - 2u_2 = 0 \Rightarrow \boxed{u_2 = \frac{x_2}{2}}$$

$$\Rightarrow f_2^*(x_2) = \frac{x_2}{2}\left(x_2 - \frac{x_2}{2}\right) = \frac{x_2^2}{4}$$

Since $u_1 = x_2 - u_2 = x_2 - \frac{x_2}{2} = \frac{x_2}{2}$

$$\Rightarrow \boxed{u_1 = \frac{x_2}{2}}$$

From stage 3: $f_3(x_3) = \underset{0 \leq u_3 \leq 10}{\text{Max}} \{u_3 f_2^*(x_2)\} = \underset{0 \leq u_3 \leq 10}{\text{Max}} \left\{u_3 \frac{x_2^2}{4}\right\}$

$$, \quad x_2 = x_3 - u_3$$

إذا النقطة

$$\Rightarrow \quad f_3(x_3) = \underset{0 \le u_3 \le 10}{Max} \left\{ u_3 \frac{(x_3-u_3)^2}{4} \right\}$$

Let $\psi = \frac{u_3}{4}(x_3-u_3)^2$

$$\frac{\partial \psi}{\partial u_3} = \frac{1}{4}(x_3-u_3)^2 - \frac{u_3}{2}(x_3-u_3) = 0$$

$$\Rightarrow \quad \frac{(x_3-u_3)}{2}\left[\frac{x_3-u_3}{2} - u_3\right] = 0$$

$x_3 - u_3 = 0$ or $\frac{x_3-u_3}{2} - u_3 = 0$

$\boxed{u_3 = x_3}$ $\Downarrow$ $\boxed{u_3 = \frac{1}{3}x_3}$

Case 1: if $u_3 = x_3$, $x_3 = 10 \Rightarrow u_3 = x_3 = 10$

$$\Rightarrow x_2 = x_3 - u_3 = 0 \ , \ u_2 = \frac{x_2}{2} = \frac{0}{2} = 0$$

$$u_1 = \frac{x_2}{2} = \frac{0}{2} = 0$$

$$\Rightarrow u_1 u_2 u_3 = \prod_{i=1}^{3} u_i = 10(0)(0) = 0$$

Case 2: if $u_3 = \frac{x_3}{3}$, $x_3 = 10 \Rightarrow \boxed{u_3 = 10/3}$

$$\Rightarrow x_2 = x_3 - u_3 = 10 - \frac{10}{3} = \frac{20}{3}$$

$$\Rightarrow u_2 = \frac{x_2}{2} = \frac{10}{3}$$

$$u_1 = \frac{x_2}{2} = \frac{10}{3}$$

$$\Rightarrow u_1 = u_2 = u_3 = \frac{10}{3} \Rightarrow Max \prod_{i=1}^{3} u_i = \frac{1000}{27}$$

In information theory, the expected amount of information is measured by Shannon – Wiener measure (or Entropy function) given by

$$H(P_1, P_2, \cdots, P_n) = -\sum_{i=1}^{n} P_i \cdot \log_2 P_i$$

where $P_i \geq 0$, $\sum_{i=1}^{n} P_i = 1 \quad \forall\ i = 1, 2, \cdots, n$

Use the dynamic programing to solve the following problem:

$$\text{Max } H(P_1, P_2, \cdots, P_n)$$

$$\text{s.t. } \sum_{i=1}^{n} P_i = 1 ;$$

$$P_i \geq 0 \quad \forall\ i = 1, 2, \cdots, n$$

Solution: This is an n-stage problem. Since Max H = Min $-H$

$\Rightarrow$ Then the problem can be written as:

$$\text{Min } \sum_{i=1}^{n} P_i \log_2 P_i$$

$$\text{s.t. } \sum_{i=1}^{n} P_i = 1 \quad ; \quad P_i \geq 0 \quad \forall\ i = 1, 2, \cdots, n$$

stage 1 ($n=1$): $S_1 = P_1 = 1 \Rightarrow f_1^*(S_1) = f_1(1) = \min P_1 \log_2 P_1$

$$\Rightarrow \boxed{f_1^*(1) = 1 \log_2 1}$$

stage 2 ($n=2$): $\boxed{S_2 = P_1 + P_2 = 1}$ let $P_2 = z \Rightarrow P_1 = 1 - z$

$$f_2(S_2) = \min(P_1 \log_2 P_1 + P_2 \log_2 P_2)$$

$$f_2(S_2) = \min_{0 \leq z \leq S_2} \left((1-z) \log_2 (1-z) + z \log_2 z \right)$$

$$= \min_{0 \leq z \leq 1} \left[(1-z) \log_2 (1-z) + z \log_2 z \right]$$

Let $\alpha(z) = (1-z) \log_2 (1-z) + z \log_2 z$

$$\alpha'(z) = (1-z) \frac{-1}{(1-z)\ln 2} + \log_2 (1-z) \cdot (-1) + z \left(\frac{1}{z} \frac{1}{\ln 2} \right) + \log_2 z$$

$$= \cancel{\frac{-1}{\ln 2}} + (-1) \log_2 (1-z) + \cancel{\frac{1}{\ln 2}} + \log_2 z$$

Setting $\alpha'(z) = 0 \Rightarrow \log_2 z - \log_2 (1-z) = 0 \Rightarrow \log_2 \frac{z}{1-z} = 0$

$$\Rightarrow 2^0 = \frac{z}{1-z} \Rightarrow 1 - z = z \Rightarrow z = 1/2 \in [0,1]$$

$$\Rightarrow P_1 = P_2 = 1/2$$

$$\therefore \quad f_2^*(s_2) = \frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2} = 2\left(\frac{1}{2}\log_2\frac{1}{2}\right)$$

$$\Rightarrow \boxed{f_2^*(s_2) = 2\left(\frac{1}{2}\log_2\frac{1}{2}\right)} \quad (*)$$

stage 3 $(n=3)$: $s_3 = P_1 + P_2 + P_3 = 1$ let $P_3 = z$ and $P_1 + P_2 = 1 - z$

$$f_3(s_3) = \min_{0 \le z \le s_3} \left(P_3\log_2 P_3 + \underbrace{P_2\log_2 P_2 + P_1\log_2 P_1}\right)$$

$$= \min_{0 \le z \le 1} \left(z\log_2 z + f_2^*(s_2)\right)$$

Since from stage 2 we have $s_2 = P_1 + P_2 \Rightarrow s_2 = 1 - z$

$$f_3(s_3) = \min_{0 \le z \le 1} \left(z\log_2 z + f_2^*(1-z)\right)$$

from $(*)$ we have $f_2^*(1-z) = 2\left(\frac{1-z}{2}\log\left(\frac{1-z}{2}\right)\right)$

$$\Rightarrow f_3(s_3) = \min_{0 \le z \le 1}\left[z\log_2 z + (1-z)\log_2\left(\frac{1-z}{2}\right)\right]$$

Set $\psi(z) = z\log_2 z + (1-z)\log_2\left(\frac{1-z}{2}\right)$

$$\psi'(z) = \cancel{z}\,\frac{1}{\cancel{z}}\frac{1}{\ln 2} + \log_2 z + (1-\cancel{z})\,\frac{\cancel{2}\cdot}{\cancel{1-z}}\,\frac{1}{\ln 2}\left(\frac{-1}{2}\right) - \log_2\left(\frac{1-z}{2}\right)$$

$$\Rightarrow \psi'(z) = \log_2(z) - \log_2\left(\frac{1-z}{2}\right)$$

setting $\psi'(z) = 0 \Rightarrow \log_2 z - \log_2\left(\frac{1-z}{2}\right) = 0 \Rightarrow \frac{2z}{1-z} = 1$

$$\Rightarrow \boxed{z = 1/3}$$

$$\therefore \quad P_3 = z = 1/3, \quad P_1 + P_2 = 1 - 1/3 = 2/3$$

from stage 2 $\Rightarrow P_1 = P_2 \Rightarrow 2P_1 = 2/3 \Rightarrow P_1 = P_2 = 1/3$

$$\Rightarrow \quad f_3^*(s_3) = \frac{1}{3}\log_2\frac{1}{3} + \frac{2}{3}\log_2\frac{1}{3} = 3\left(\frac{1}{3}\log_2\frac{1}{3}\right)$$

In general, $\quad P_1 = P_2 = \dots = P_n = \frac{1}{n}$ and $H_{min} = n\left(\frac{1}{n}\log_2\frac{1}{n}\right)$

$$\Rightarrow \quad H_{min} = \log_2(1/n) \Rightarrow H_{max} = -\log_2\left(\frac{1}{n}\right)$$

$$\Rightarrow \quad H_{max} = -\left[\log_2 1 - \log_2 n\right]$$

$$H_{max} = \log_2 n$$

(7) Use Dynamic programming to solve the following l.p.p. :

$$\text{Max } z = 4x_1 + 14x_2 ,$$
$$\text{s.t.} \quad 2x_1 + 7x_2 \le 21 ,$$
$$7x_1 + 2x_2 \le 21 ,$$
$$x_1, x_2 \ge 0$$

Solution: We have $b_1 = 21$, $b_2 = 21$

Stage1 : $f_1(b_1, b_2) = \max\limits_{0 \le x_1 \le b} [4x_1]$, $b = \min\left\{\frac{21}{2}, \frac{21}{7}\right\} = 3$

$$= \max_{0 \le x_1 \le 3} [4x_1]$$

from the Constraints: $\quad x_1 \le \dfrac{21 - 7x_2}{2}$, $x_1 \le \dfrac{21 - 2x_2}{7}$

$$\Rightarrow \quad f_1(b_1, b_2) = 4 \max_{0 \le x_1 \le 3} [x_1]$$

$$\Rightarrow \quad f_1^*(b_1, b_2) = 4x_1^* , \quad x_1^* = \min\left(\frac{21 - 7x_2}{2}, \frac{21 - 2x_2}{7}\right)$$

stage2 : $f_2(b_1, b_2) = \max\limits_{0 \le x_2 \le b}\left(14x_2 + 4x_1^*\right)$, $b = \min\left\{\frac{21}{7}, \frac{21}{2}\right\} = 3$

$$\Rightarrow \quad f_2(b_1, b_2) = \max_{0 \le x_2 \le 3}\left(14x_2 + 4\min_{0 \le x_2 \le 3}\left(\frac{21 - 7x_2}{2}, \frac{21 - 2x_2}{7}\right)\right)$$

$$\min_{0 \le x_2 \le 3} \left(\frac{21-7x_2}{2}, \frac{21-2x_2}{7} \right) = \begin{cases} \frac{21-2x_2}{7} ; & 0 \le x_2 < 7/3 \\ \frac{21-7x_2}{2} ; & \frac{7}{3} \le x_2 \le 3 \end{cases}$$

$$\Rightarrow f_2(b_1, b_1) = \max_{0 \le x_2 \le 3} \begin{bmatrix} 14x_2 + \frac{4}{7}(21-2x_2) ; & 0 \le x_2 < 7/3 \\ 14x_2 + 2(21-7x_2) ; & 7/3 \le x_2 \le 3 \end{bmatrix}$$

At $x_2 = 7/3 \Rightarrow 14\left(\frac{7}{3}\right) + \frac{4}{7}\left(21 - \frac{14}{3}\right) = 42$

$\Rightarrow 14\left(\frac{7}{3}\right) + 2\left(21 - 7\left(\frac{7}{3}\right)\right) = 42$

At $x_2 = 3 \Rightarrow$

$14(3) + 2(21-21) = 42$

$\Rightarrow f_2^*(b_1, b_1) = 42$ at $x_2^* = 3, 7/3$

Since $x_1^* = \min\left(\frac{21-7x_2}{2}, \frac{21-2x_2}{7} \right)$

then

at $x_2 = \frac{7}{3} \Rightarrow x_1^* = \min\left(\frac{21-7(7/3)}{2}, \frac{21-2(7/3)}{7} \right) = 7/3$

at $x_2 = 3 \Rightarrow x_1^* = \min\left(\frac{21-21}{2}, \frac{21-6}{7} \right) = 0$

So the optimum value of $x_1^* = 0$ and hence the optimum solution is $(0, 3)$ and $Z_{max} = f_2^*(b_1, b_2) = 42$