KING SAUD UNIVERSITY DEPARTMENT OF MATHEMATICS
M316.TIME 3H, FULL MARKS 40, FINAL EXAM S2-2025/26

Question 1[3,3,5]. **a)** Assume that the function f satisfies $|f(x)| \leq C$ for all $x \in [0, \infty)$ and $\int\limits_0^\infty |f(x)|\, dx < \infty$. Show that $f \in \mathfrak{L}^2(0, \infty)$. Show that the converse is not true by giving an example of a bounded function in$\mathfrak{L}^2(0, \infty)$ which is not integrable on $(0, \infty)$.

b) Obtain the limit of the sequence of functions $F_n(x) = nx(1-x)^n$ in the space $\mathfrak{L}^2(0, 1)$.

c) For which value of λ, the following functions belongs to $\mathfrak{L}^2(0, \infty)$:
i) $f(x) = e^{(1-5\lambda)x}$, **ii)** $g(x) = e^{\lambda x}\sqrt{1 - \cos x}$, **iii)** $h(x) = x^{1/2}e^{2\lambda x}$.

Question 2[5,4,4]. **a)** Determine the Fourier transform for the function

$$f(x) = e^{-a|x|}, a > 0,$$

and deduce the value of the integral $\int\limits_{-\infty}^{\infty} \frac{d\xi}{a^2+\xi^2}$.

b) Let

$$f(x) = \begin{cases} 3, & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

i) Find the Fourier cosine integral of f

ii) Deduce the value of the integral $\int\limits_{-\infty}^{\infty} \frac{\sin \xi}{\xi} d\xi$.

c) Solve the integral equation

$$\int_0^\infty f(x)\sin(x\xi)dx = \begin{cases} 2, & 0 < \xi < 1 \\ e^{-\xi}, & \xi > 1 \end{cases}$$

Question 3[5,3]. **a)** Find the Fourier series for the function

$$g(x) = 1 - \frac{2}{\pi}x \quad \text{such that } g(x + 2\pi) = g(x)$$

b) Deduce that $\sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

Question 4[3,3,2]. Consider the eigenvalue problem

$$\begin{cases} x^2u'' + xu' + \lambda u = 0, \; x \in [1, 2] \\ u(1) = 0, \; u(2) = 0. \end{cases} \tag{1}$$

i) Write the equation in problem (1) in the Sturm-Liouville form.

ii) Show that

$$\int_1^2 \left[\lambda \frac{u^2}{x} - x\left(\frac{du}{dx}\right)^2\right] dx = 0. \tag{2}$$

(Hint: Multiply the Sturm-Liouville from by u and integrate over the interval $[1, 2]$).

iii) Deduce from (2) that all eigenvalues are positive.