College of Sciences
Department of Mathematics

Final Exam

Math 508

Exercise 1

1. Consider the ODE $u_t + cu_x = 0$, where c is a constant.
   (a) Show that $u = \sin(x - ct), u = \cos(x - ct)$ and $u = 5(x - ct)^2$ are solutions.
   (b) Show that $u = 7\sin(x-ct), u = 3\cos(x-ct)$ and $u = 7\sin(x-ct)-3\cos(x-ct)$ also are solutions.
2. Consider the ODE $u_x^2 + u_y^2 = 1$.
   (a) Show that $u = x$ and $u = y$ are solutions.
   (b) Are $u = 3x$ and $u = -4y$ solutions ?
   (c) Is $u = x + y$ a solution ?
   (d) Find all solutions of the form $u = ax + by$, where a and b are constants.

Exercise 2

Determine whether the ODE is linear or nonlinear, and prove your result. If it is nonlinear, point out the term or terms which make it nonlinear.

1. $u_{xy} + 5u = x^2 y$
2. $u_{xx} + uu_{xy} = 1$
3. $y^2 u_{xx} + u_{yy} = \cos(x)$.
4. $u_{xxy} - \sin(x) u_{yy} + x - y = 0$.
5. $u_x + e^u u_y = 2x + 1$.

Exercise 3

We consider the following Cauchy problem

$$\begin{cases} y'(t) = f(t, y(t)), \ t \in [0, T] \\ \\ y(0) = y_0, \end{cases}$$

where $y_0 \in \mathbb{R}$. We suppose that there exists $L > 0$ such that for all $t \in [0, T]$ and for all $x, y \in \mathbb{R}$,

$$|f(t, y) - f(t, x)| \leq L|x - y|.$$

We consider the following midpoint method :

$$\begin{cases} \hat{y} = y_n + \frac{h}{2} f(t_n, y_n) \\ \\ y_{n+1} = y_n + h f(t_n + \frac{h}{2}, \hat{y}) \\ \\ t_{n+1} - t_n = h, \quad n \geq 0. \end{cases}$$

1. Put the midpoint scheme in the form $y_{n+1} = y_n + h\phi(t_n, y_n, h)$, where ϕ is to be determined.
2. Prove that the method is consistent.
3. Prove that the method is stable.
4. Deduce that the method is convergent.

Exercise 4

We consider the following differential equation

$$y'(t) = y(t) + e^{2t},$$

with $y(0) = 2$.

We consider the following modified Euler method

$$\begin{cases} \hat{y} = y_n + h f(t_n, y_n) \\ \\ y_{n+1} = y_n + \frac{h}{2} \left[f(t_n, y_n) + f(t_{n+1}, \hat{y}) \right] \\ \\ t_{n+1} - t_n = h, \quad n \geq 0. \end{cases}$$

1. Prove that $y(t) = e^t + e^{2t}$ is the solution of the above differential equation.
2. Take $h = 0,1$
   (a) Do 3 iterations of the modified Euler method.

(b) Compute the error on y_3 by comparing the results with the solution $y(0,3)$.

3. Take $h = 0,05$

   (a) Do 6 iterations of the modified Euler method.

   (b) Compute the error on y_6 by comparing the results with the solution $y(0,3)$.