**PROBLEM 1** : Consider the unconstrained problem :

$$\min\,(\max)\; f(x_1, x_2) = x_1^3 + x_2^4 - 2x_1^2 - 8x_2^2 + 10$$

**1.** Find all critical points and determine their kind.

**2.** Repeat questions 1. for the following function : (أعد السؤال ١. للدالة التالية)

$$f(x_1, x_2, x_3) = \frac{x_1^2}{2} + x_2^3 + x_3^2 - x_1x_2 - 2x_2x_3 + 2x_1 - 2x_2$$

**PROBLEM 2** : **1.** Consider the constrained problem :

$$\min z = f(x_1, x_2) = -3x_1 + x_2$$

$$s.t.: \ (x_1 - 1)^3 - x_2 = 0$$

**a.** Use Lagrangean technique to determine all critical points.

**b.** Draw the feasible region and the levels curves of $f(x_1, x_2)$ for $f = 8,\ 10$.

From the figure, give the value of the optimal solution $x_1^*$ and $x_2^*$. Compare the result with the critical points obtained in part 1.

**2.** Repeat questions a. and b. for the following constrained problem :

$$\min z = f(x_1, x_2) = 2x_1 + x_2 + 1$$

$$s.t.: \ (x_1 - 3)^2 + x_2^2 \le 1$$

$$x_1 - x_2 = 3$$

**PROBLEM 3** : Consider the constrained problem : $\min z = f(x_1, x_2) = (x_1 - 6)^2 + (x_2 - 6)^2$

$$s.t.: \ x_1 + x_2 \ge 3$$

$$(x_1 - 3)^2 + x_2^2 \le 9$$

$$x_1^2 + x_2^2 \le 9$$

Write the Lagrangean optimality conditions (DO NOT SOLVE).

**PROBLEM 4** : **1.** Consider the univariate unconstrained optimization problem :

$$Minimize\ f(x) = 5x^2 - 8x + 2 =$$

Use Dichotomous search technique to locate the maximum of $f(x)$ in $[a_1, b_1] = [1, 4]$ ( Take: $\epsilon = 0.1$). Give an approximate value to $x^*$.**(five iterations up to $[a_5, b_5]$ خمس تكرارات إلى)**

**2.** Consider the univariate unconstrained optimization problem :

$$Minimize\ f(x) = \begin{cases} \dfrac{1}{x}, & 1 \le x \le 2 \\ x^2, & 2 \le x \le 4 \end{cases}$$

Use Fibonacci search technique to locate the minimum of $f(x)$ in $[a_1, b_1] = [1, 4]$ ( Take: $l = 0.5$, $\varepsilon = 0.1$). Give an approximate value to $x^*$.

OR 231

# Final exam

# = Model answers =

## Problem 1:

(1) $f(x_1, x_2) = x_1^3 + x_2^4 - 2x_1^2 - 8x_2^2 + 10$

$$\nabla f(x_1, x_2) = \begin{bmatrix} 3x_1^2 - 4x_1 \\ 4x_2^3 - 16x_2 \end{bmatrix}$$

$3x_1^2 - 4x_1 = 0$

$x_1(3x_1 - 4) = 0$

$x_1 = 0$ or $x_1 = \frac{4}{3}$

$4x_2^3 - 16x_2 = 0$

$4x_2(x_2^2 - 4) = 0$

$x_2 = 0$ , $x_2 = \pm 2$

critical points:

$(0,0)$ , $(0,2)$ , $(0,-2)$

$(\frac{4}{3}, 0)$ , $(\frac{4}{3}, 2)$ , $(\frac{4}{3}, -2)$

$$H(x_1, x_2) = \nabla^2 f(x_1, x_2) = \begin{bmatrix} 6x_1 - 4 & 0 \\ 0 & 12x_2 - 16 \end{bmatrix}$$

→ $(0,0)$ $H = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix}$ $H_1 = -4$ , $H_2 = 64$

local max. -ve definite

→ $(0,2)$ $H = \begin{bmatrix} -4 & 0 \\ 0 & 8 \end{bmatrix}$ $H_1 = -4$ , $H_2 = -32$

saddle point indefinite

→ $(0,-2)$ $H = \begin{bmatrix} -4 & 0 \\ 0 & -40 \end{bmatrix}$ $H_1 = -4$ , $H_2 = 160$

local max. -ve definite

→ $(\frac{4}{3}, 0)$ $H = \begin{bmatrix} 4 & 0 \\ 0 & -16 \end{bmatrix}$ $H_1 = 4$ , $H_2 = -64$

saddle point indefinite

→ $(\frac{4}{3}, 2)$ $H = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$ $H_1 = 4$ , $H_2 = 32$

local min. +ve definite

→ $(\frac{4}{3}, -2)$ $H = \begin{bmatrix} 4 & 0 \\ 0 & -40 \end{bmatrix}$ $H_1 = 4$ , $H_2 = -160$

saddle point indefinite

(2) $f(x_1, x_2, x_3) = \frac{x_1^2}{2} + x_2^3 + x_3^2 - x_1 x_2 - 2x_2 x_3 + 2x_1 - 2x_2$

$$\nabla f(X) = \begin{bmatrix} x_1 - x_2 + 2 \\ 3x_2^2 - x_1 - 2x_3 - 2 \\ 2x_3 - 2x_2 \end{bmatrix}$$

$x_1 - x_2 + 2 = 0 \Rightarrow \underline{x_1 = x_2 - 2}$

$2x_3 - 2x_2 = 0 \Rightarrow \underline{x_3 = x_2}$

$3x_2^2 - x_1 - 2x_3 - 2 = 0$

$3x_2^2 - x_2 + 2 - 2x_2 - 2 = 0$

$3x_2^2 - 3x_2 = 0 \qquad 3x_2(x_2 - 1) = 0$

$x_2 = 0, \quad x_2 = 1$

$x_1 = -2, \quad x_1 = -1$

$x_3 = 0, \quad x_3 = 1$

critical points:

$(-2,0,0)$ & $(-1,1,1)$

$$\nabla^2 f(X) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6x_2 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

→ $(-2,0,0)$ saddle point

$$H = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$H_1 = 1$

$H_2 = -1$

$H_3 = -4 - 2 = -6$

indefinite

→ $(-1,1,1)$ local min.

$$H = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$H_1 = 1$

$H_2 = 5$

$H_3 = 8 - 2 = 6$

+ve definite

Problem 2

1 $\quad \min\ z = f(x_1, x_2) = -3x_1 + x_2$

$\quad$ s.t. $\quad (x_1 - 1)^3 - x_2 = 0$

a- Using Lagrange :

$$L(X, \lambda) = -3x_1 + x_2 + \lambda\left[(x_1-1)^3 - x_2\right]$$

$$\nabla L(X, \lambda) = 0$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow -3 + 3\lambda(x_1-1)^2 = 0$$

$$\lambda = \frac{1}{(x_1-1)^2}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 1 + (-\lambda) = 0 \Rightarrow \underline{\underline{\lambda = 1}}$$

$$(x_1-1)^2 = 1$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow (x_1-1)^3 - x_2 = 0$$

$x_1 = 2 \rightarrow \underline{\underline{x_2 = 1}}$ or $x_1 = 0 \rightarrow \underline{\underline{x_2 = -1}}$

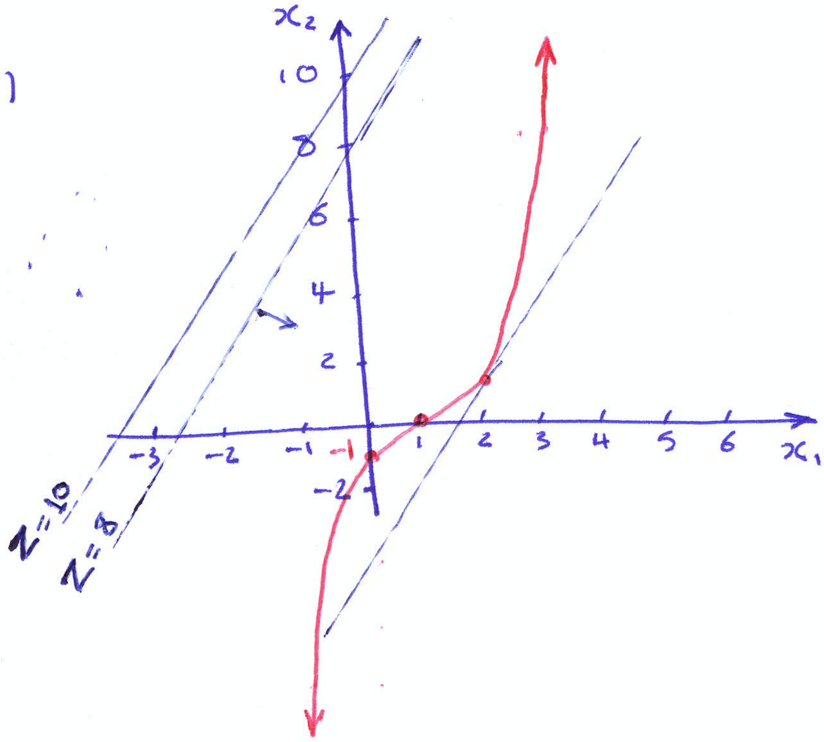
critical points are: $(2, 1)$ , $(0, -1)$

b- $\min z$ at $(2, -1)$

$\min z = -6 + 1 = \underline{\underline{-5}}$

$x_1^* = 2$ , $x_2^* = -1$

critical point from Lagrange



(2) $\min\ z = f(x_1, x_2) = 2x_1 + x_2 + 1$

s.t. $(x_1 - 3)^2 + x_2^2 \leq 1$

$x_1 - x_2 = 3$

→ Using Lagrange:

$$L(x, \lambda) = 2x_1 + x_2 + 1 + \lambda_1((x_1-3)^2 + x_2^2 + s^2 - 1) + \lambda_2(x_1 - x_2 - 3)$$

$$\frac{\partial L}{\partial x_1} = 2 + 2\lambda_1(x_1 - 3) + \lambda_2 = 0 \quad (1)$$

$$\frac{\partial L}{\partial x_2} = 1 + 2\lambda_1 x_2 - \lambda_2 = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda_1} = (x_1-3)^2 + x_2^2 + s^2 - 1 = 0 \quad (3)$$

$$\frac{\partial L}{\partial \lambda_2} = x_1 - x_2 - 3 = 0 \quad (4) \Rightarrow \underline{\underline{x_1 = x_2 + 3}}$$

$$\frac{\partial L}{\partial s} = 2\lambda_1 s = 0 \quad (5) \qquad \lambda_1 = 0 \quad \text{or} \quad s = 0$$

$\lambda_2 = -2$ & $\lambda_2 = 1$ ✗

if $\lambda_1 = 0$

from (1) $\underline{\underline{\lambda_2 = -2}}$ from (2) $\lambda_2 = 1$ $\underline{\underline{\text{refused}}}$

if $s = 0$

from (4) $x_1 = x_2 + 3$

subs. in (3) $(x_2 + 3 - 3)^2 + x_2^2 + 0 - 1 = 0$

$2x_2^2 = 1 \qquad x_2^2 = \frac{1}{2} \qquad \underline{\underline{x_2 = \pm\frac{1}{\sqrt{2}}}}$

$\underline{\underline{x_1 = 3 \pm \frac{1}{\sqrt{2}}}}$

critical points:

$\left(\frac{1}{\sqrt{2}}, 3 + \frac{1}{\sqrt{2}}\right)$ & $\left(-\frac{1}{\sqrt{2}}, 3 - \frac{1}{\sqrt{2}}\right)$

→ Drawing feasible region:

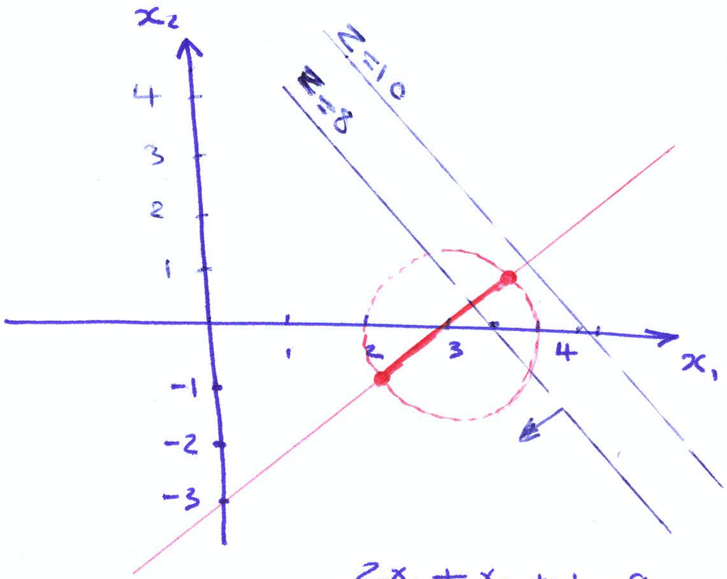
$$x_1 - x_2 = 3$$

$$\underline{\underline{x_1 = x_2 + 3}}$$

$$(x_1 - 3)^2 + x_2^2 = 1$$

$$2x_2^2 = 1 \qquad x_2 = \pm \frac{1}{\sqrt{2}}$$

$$x_1 = 3 \pm \frac{1}{\sqrt{2}}$$




$$2x_1 + x_2 + 1 = 8$$

$$\underline{2x_1 + x_2 = 7}$$

$$2x_1 + x_2 + 1 = 10$$

$$\underline{2x_1 + x_2 = 9}$$

Min. Z at $\left(3 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$$x_1^* = 3 - \frac{1}{\sqrt{2}}, \qquad x_2^* = -\frac{1}{\sqrt{2}}$$

$$\min Z = 2\left(3 - \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} + 1 = 6 - \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 1$$

$$= 7 - \frac{3}{\sqrt{2}} = \underline{\underline{4.88}}$$

3a

problem 3

$$\min z = f(x_1, x_2) = (x_1-6)^2 + (x_2-6)^2$$

s.t.

$$x_1 + x_2 \geq 3$$

$$(x_1-3)^2 + (x_2)^2 \leq 9$$

$$x_1^2 + x_2^2 \leq 9$$

$$L(x, \lambda, S) = (x_1-6)^2 + (x_2-6)^2 + \lambda_1 (x_1 + x_2 - S_1^2 - 3) + \lambda_2 [(x_1-3)^2 + x_2^2 + S_2^2 - 9] + \lambda_3 [x_1^2 + x_2^2 + S_3^2 - 9]$$

$$\frac{\partial L}{\partial x_1} = 2(x_1-6) + \lambda_1 + 2\lambda_2(x_1-3) + 2\lambda_3 x_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 2(x_2-6) + \lambda_1 + 2\lambda_2 x_2 + 2\lambda_3 x_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = x_1 + x_2 - S_1^2 - 3 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = (x_1-3)^2 + x_2^2 + S_2^2 - 9 = 0$$

$$\frac{\partial L}{\partial \lambda_3} = x_1^2 + x_2^2 + S_3^2 - 9 = 0$$

$$\frac{\partial L}{\partial S_1} = -2\lambda_1 S_1 = 0 \qquad \lambda_1 S_1 = 0$$

$$\frac{\partial L}{\partial S_2} = 2\lambda_2 S_2 = 0 \qquad \lambda_2 S_2 = 0$$

$$\frac{\partial L}{\partial S_3} = 2\lambda_3 S_3 = 0 \qquad \lambda_3 S_3 = 0$$

problem 4

Dichotomous search method:

$$\text{Min } f(x) = 5x^2 - 8x + 2 \qquad [a_1, b_1] = [1, 4]$$

$$1 \leq x \leq 4 \qquad \text{take } \varepsilon = 0.1$$

| K | $a_K$ | $b_K$ | $x_K$ | $y_K$ | $f(x_K)$ | $f(y_K)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 4 | 2.4 | 2.6 | 11.6 | 15 |
| 2 | 1 | 2.6 | 1.7 | 1.9 | 2.85 | 4.85 |
| 3 | 1 | 1.9 | 1.35 | 1.55 | 0.3125 | 1.6125 |
| 4 | 1 | 1.55 | | | | |

if $b_K - a_K < l$ stop.

1- else: $x_K = \dfrac{a_K + b_K}{2} - \varepsilon \qquad f(x_K)$

$y_K = \dfrac{a_K + b_K}{2} + \varepsilon \qquad f(y_K)$

2- if $f(x_K) < f(y_K)$

$a_{K+1} = a_K \quad , \quad b_{K+1} = y_K$

else $a_{K+1} = x_K \quad , \quad b_{K+1} = b_K$

goto step 1