مع الإجابة الكاملة

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Final exam M.318
Second semester 2023
Course Instructor: **Prof.M. DAMLAKHI.**
Course Title: Math 318 (Differential Equations)
Date: Thursday 25/7/1444 – 16/2/2023
Time: (8-11) am, 3 Hours.

Question1(3+3+4).

a) Find the solution of the differential equation:

$$(3) \qquad y' - \frac{5}{x}y = -xy^3 \ , \ \ x > 0.$$

b) Solve the differential equation:

$$(3) \qquad x^{-1}y' - 2x^{-2}y = x\cos x \, , \qquad x > 0.$$

(4) **c)** A hot iron rod was left in a room where the temperature was $20^0\,C$.After one minute the temperature of the rod was recorded $35^0 C$, and after two minutes it was $27.5^0 C$. What was the initial temperature of the rod?

Question2(5+5).

a) Solve the following system of **ODE** by **Elimination** method:

$$(5) \qquad \begin{aligned} x'(t) &= 3x - 4y + 1 \\ y'(t) &= 4x - 7y + 10t \end{aligned}$$

b) Find the general solution on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ of the differential equation (Hint: use **Superposition principle)**

$$(5) \qquad y'' + y = tanx + 3x - 1.$$

Question3(4+6).

a) Find the general solution of the differential equation:

(4) $\quad x\,y'' - 2(x+1)y' + (x+2)y = 0$, where $y_1 = e^x$ is a particular solution of the differential equation and $x > 0$.

b) Solve the following differential equation by using the power series method about the **ordinary** point $x_0 = 1$(**Hint**: find the first 4 terms only).

$$(6) \qquad y'' - 2(x-1)y' + 2y = 0$$

Question4(2+4+4).

a) **Show** that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$. 2

b) **Evaluate** $\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$. 4

c) **Solve** the following **IVP** by using Laplace Transform method.

$$y''(t) - 2y'(t) + 5y(t) = -8e^{-t}, \quad y(0) = 2, \quad y'(0) = 12$$

(Hint: Use part **a & b)).** 4

Complete solution of final exam 14.318

Second Semester 1444-1445 (2023)

Question ①

ⓐ $y' - \frac{5}{x}y = -xy^3; \quad x>0, \; y \neq 0$

(2) $y^{-3}y' - \frac{5}{x}y^{-2} = -x, \quad u = y^{-2}, \quad u' = -2y' y^{-3}, \quad -\frac{u'}{2} = y'y^{-3}$

$-\frac{u'}{2} - \frac{5}{x}u = -x, \Rightarrow u' + \frac{10}{x}u = 2x, \quad \mu(x) = e^{\int \frac{10}{x}dx} = e^{\ln x^{10}} = x^{10}$

(1) $u x^{10} = \int 2x \, x^{10} dx = 2\int x^{11} dx$

$y^{-2}x^{10} = \frac{1}{6}x^{12} + c$ or $x^{10} = y^2\left(\frac{1}{6}x^{12} + c\right)$ is the solution of the D.E.

ⓑ $x^{-1}y' - 2x^{-2}y = x\cos x; \quad x>0$

$y' - 2x^{-1}y = x^2\cos x$ is Linear D.E

(2) $\mu(x) = e^{-2\int \frac{dx}{x}} = e^{\ln x^{-2}} = \frac{1}{x^2}$

(1) $y\frac{1}{x^2} = \int x^2\cos x \cdot \frac{1}{x^2}dx = \int \cos x\, dx = \sin x + c$

$\boxed{y = x^2(\sin x + c)}$ is the general solution of the D.E.

ⓒ

(1) $T(t) = T_s + ce^{kt}, \quad T_s = 20°c, \quad T(1) = 35°c, \quad T(2) = 27.5°c$

$T(t) = 20 + ce^{kt}, \quad \begin{cases} T(1) = 35 = 20 + ce^{k} & c \neq 0 \\ T(2) = 27.5 = 20 + ce^{2k} \end{cases}$

$\begin{cases} 15 = ce^{k} \\ 7.5 = ce^{2k} \end{cases} \Rightarrow \frac{15}{7.5} = \frac{1}{e^{k}} \Rightarrow 2 = e^{-k}, \quad \boxed{k = -\ln 2}$ (1)

$15 = ce^{-\ln 2} = ce^{\ln 2^{-1}} = \frac{c}{2} \Rightarrow \boxed{c = 30}$ (1)

(1) $T(t) = 20 + 30e^{(-\ln 2)t}$, then $\boxed{T(0) = 20 + 30 = 50°c}$

Question ②

ⓐ $x'(t) = 3x - 4y + 1$, $\left\{ \begin{array}{l} 4((D-3)x + 4y = 1) \\ (D-3)(-4x + (D+7)y = 10t) \end{array} \right.$

$y'(t) = 4x - 7y + 10t$

$$+ \left\{ \begin{array}{l} 4(D-3) + 16y = 4 \\ -4(D-3) + (D-3)(D+7)y = (D-3)(10t) \end{array} \right.$$

$$16y + (D^2 + 4D - 21)y = 4 + 10 - 30t$$

② $y'' + 4y' - 5y = 14 - 30t$ ...... ①

$y'' + 4y' - 5y = 0 \Rightarrow (m+5)(m-1) = 0$, $m = -5, 1$

$y_c = c_1 e^{-5t} + c_2 e^t$

$y_p = A + Bt$, $y_p' = B$, $y_p'' = 0 \Rightarrow 0 + 4B - 5A - 5Bt = 14 - 30$

$\left\{ \begin{array}{l} 4B - 5A = 14 \\ -5B = -30 \end{array} \right. \Rightarrow B = 6$, $24 - 5A = 14$, $+5A = 10$

$A = 2$

① $y_p = 2 + 6t \Rightarrow y = y_c + y_p = \boxed{c_1 e^{-5t} + c_2 e^t + 2 + 6t}$

is the G. solution of the D.E. ①

$4x = y'(t) + 7y - 10t$

$4x = -5c_1 e^{-5t} + c_2 e^t + 6 + 7c_1 e^{-5t} + 7c_2 e^t + 14 + 42t - 10t$

$4x = 2c_1 e^{-5t} + 8c_2 e^t + 20 + 32t$

② $\boxed{x(t) = \frac{1}{2} c_1 e^{-5t} + 2c_2 e^t + 5 + 8t}$

ⓑ $y'' + y = \tan x + 3x - 1$

1) $y'' + y = 0$, $m^2 + 1 = 0$, $m = \mp i$ $y_c = c_1 \cos x + c_2 \sin x$ ①

$y'' + y = 3x - 1$, $y_{1p} = A + Bt$, $y_{1p}' = B$, $y_{1p}'' = 0$

$y_{1p}'' + y_{1p} = 0 + A + Bt = 3x - 1 \Rightarrow A = -1$, $B = 3$

② $\boxed{y_{1p} = 3x - 1}$

2) $y'' + y = \tan x$ $y_{2p} = u_1 \cos x + u_2 \sin x$, $y_1 = \cos x$, $y_2 = \sin x$

$\left\{ \begin{array}{l} u_1' \cos x + u_2' \sin x = 0 \\ -u_1' \sin x + u_2' \cos x = \tan x \end{array} \right.$

$$W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1,\quad u_1'=\frac{\begin{vmatrix}0 & \sin x\\ \tan x & \cos x\end{vmatrix}}{W}=\frac{-\sin^2 x}{\cos x}$$

$$u_1'=-\frac{1-\cos^2 x}{\cos x}=-\sec x+\cos x$$

$$u_1=-\int \sec x\,dx+\int \cos x\,dx=\boxed{-\ln|\sec x+\tan x|+\sin x=u_1}$$

$$u_2'=\frac{\begin{vmatrix}\cos x & 0\\ -\sin x & \tan x\end{vmatrix}}{W}=\sin x,\quad u_2=\int \sin x\,dx=-\cos x$$

Then $y_{2p}=-\cos x\ln|\sec x+\tan x|+\cos x\sin x-\sin x\cos x$

2 $\boxed{y_{2p}=-\cos x\ln|\sec x+\tan x|}$

1 So $\boxed{y_p=y_{1p}+y_{2p}=-\cos x\ln|\sec x+\tan x|+3x-1}$

Then the general solution of the D.E is

$$\boxed{y=c_1\cos x+c_2\sin x+y_p}$$

Question 3 $\quad x y''-2(x+1)y'+(x+2)y=0$, $y_1=e^x$ is a solution given, $x>0$

(a)

(i) $y''-2(1+\frac{1}{x})y'+\frac{x+2}{x}y=0 \qquad P(x)=-2-\frac{2}{x}$

$$e^{-\int P(x)dx}=e^{\int(2+\frac{2}{x})dx}=e^{2x}\cdot x^2$$

1 $$y_2=y_1\int\frac{e^{-\int P(x)dx}}{y_1^2}dx=e^x\int\frac{e^{2x}x^2}{e^{2x}}dx=\frac{1}{3}e^x x^3$$

Then the general solution of the D.E is $y=c_1e^x+c_2\frac{1}{3}e^x x^3$ or

2 $\boxed{y=c_1e^x+c_3x^3e^x}$; $c_3=\frac{1}{3}c_2$.

(b) $\quad y''-2(x-1)y'+2y=0 \qquad y=\sum_0^\infty a_n(x-1)^n$; $x\in\mathbb{R}$

$$\sum_2^\infty n(n-1)a_n(x-1)^{n-2}-2(x-1)\sum_1^\infty na_n(x-1)^{n-1}+\sum_0^\infty 2a_n(x-1)^n=0$$

$$\sum_2^\infty n(n-1)a_n(x-1)^{n-2}-\sum_1^\infty 2na_n(x-1)^n+\sum_0^\infty 2a_n(x-1)^n=0 \quad 1$$

$n-2=k$ $\qquad$ $n=k$ $\qquad$ $n=k$

$n=k+2$

$\therefore \sum_{k=0}^{\infty} (k+1)(k+2) a_{k+2} (x-1)^k - \sum_{1}^{\infty} 2k a_k (x-1)^k + \sum_{0}^{\infty} 2 a_k (x-1)^k = 0$

$(2a_2 + 2a_0) + \sum_{k=0}^{\infty} \left[(k+1)(k+2) a_{k+2} - 2k a_k + 2 a_k \right] (x-1)^k = 0$ ①

$2(a_2 + a_0) = 0 \Rightarrow \boxed{a_2 = -a_0}$

$a_{k+2} = \dfrac{2(k-1) a_k}{(k+1)(k+2)}, \quad k \geq 1$

$k=1, \ a_3 = 0,$

$k=2 \quad a_4 = \dfrac{\cancel{2}(1) a_2}{3 \cdot \cancel{4}_2} = \dfrac{-1}{6} a_0$

$k=3 \quad a_5 = 0, \quad k=4, \quad a_6 = \dfrac{2(3) a_4}{5 \cdot 6} = \dfrac{-1}{30} a_0$

$- - -$ ① ① ① ①

$y = a_0 + a_1 (x-1) - (x-1)^2 a_0 + 0 - \frac{1}{6} (x-1)^4 + - -$

$= a_1 \underbrace{(x-1)}_{y_1} + a_0 \underbrace{\left[1 - (x-1)^2 - \frac{1}{6}(x-1)^4 - - - \right]}_{y_2}$

$= a_1 y_1 + a_0 y_2$, a_0 and a_1 are arbitrary constants.

Question ④

ⓐ $\mathcal{L}(e^{at}) = \int_0^{\infty} e^{at} \cdot e^{-st} \, dt = \lim_{\ell \to \infty} \int_0^{\ell} e^{-(s-a)t} \, dt \ ; \quad s > a$

②

$= \lim_{\ell \to \infty} \left[\frac{-1}{(s-a)} e^{-(s-a)\ell} + \frac{1}{s-a} \right] = \frac{1}{s-a}, \quad s > a$

ⓑ $\dfrac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)} = \dfrac{As + B}{s^2 - 2s + 5} + \dfrac{C}{s+1}$

$s+1 \qquad s^2 - 2s + 5$

$2s^2 + 10s = As^2 + Bs + As + B + Cs^2 - 2Cs + 5C$

$= (A+C)s^2 + (B + A - 2C)s + B + 5C \Big\} \Rightarrow \begin{array}{l} A + C = 2 \\ A + B - 2C = 10 \\ B + 5C = 0 \end{array}$

$A + C = 2, \quad B = -5C$

$\mp A \pm 7C = \mp 10$

$8C = -8 \Rightarrow C = -1, \quad B = 5, \quad A = 3$

$\dfrac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)} = \dfrac{3s + 5}{s^2 - 2s + 5} - \dfrac{1}{s+1}$ ②

$$\therefore \mathcal{L}^{-1}\left(\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right) = \mathcal{L}^{-1}\left(\frac{3s+5}{s^2-2s+5}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{3(s-1)+8}{(s-1)^2+4}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

2

$$= 3\mathcal{L}^{-1}\left(\frac{(s-1)}{(s-1)^2+2^2}\right) + 4\mathcal{L}^{-1}\left(\frac{2}{(s-1)^2+2^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= \boxed{3e^{+t}\cos 2t + 4e^{t}\sin 2t - e^{-t}}$$

(c) $\ddot{y}(t) - 2\dot{y}(t) + 5y(t) = -8e^{-t}, \quad y(0) = 2, \ \dot{y}(0) = 12$

$$\mathcal{L}(\ddot{y}(t)) - 2\mathcal{L}(\dot{y}(t)) + 5\mathcal{L}(y(t)) = -8\mathcal{L}(e^{-t}) = -\frac{8}{s+1}$$

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + 5Y(s) = \frac{-8}{s+1}$$

2

$$s^2Y(s) - 2s - 12 - 2sY(s) + 4 + 5Y(s) = -\frac{8}{s+1}$$

$$(s^2-2s+5)Y(s) = 2s+8 - \frac{8}{s+1} = \frac{2s^2+10s}{s+1}$$

$$Y(s) = \frac{2s^2+10s}{(s^2-2s+5)(s+1)}$$

2

$$\boxed{\mathcal{L}^{-1}(Y(s)) = y(t) = 3e^{t}\cos 2t + 4e^{t}\sin 2t - e^{-t}}$$

$$\begin{cases} x'(t) = 3x - 4y + 1 \\ y'(t) = 4x - 7y + 10t \end{cases}$$

$$\begin{cases} (D-3)x + 4y = 1 \\ -4x + (D+7)y = 10t \end{cases}$$

$$\begin{cases} (D+7)\left[(D-3)x + 4y\right] = (D+7)(1) \\ -4\left[-4x + (D+7)y\right] = -40t \end{cases}$$

$+$

$$(D+7)(D-3)x + 16x = -40t + 7$$

$$x'' + 4x' - 5x = -40t + 7 \quad \text{---} \; (1)$$

$$x = e^{mt}$$

1) $m^2 + 4m - 5 = (m+5)(m-1) = 0$

$m = -5, \; m = 1$

$$x(t) = c_1 e^{-5t} + c_2 e^{t}$$

$$x_p = A + Bt \qquad x_p' = B, \quad x_p'' = 0$$

$$x_p'' + 4x_p' - 5x_p = 0 + 4B - 5A - 5Bt = -40t + 7$$

$$4B - 5A = 7, \quad -5B = -40 \Rightarrow B = 8$$

$$32 - 5A = 7 \Rightarrow -5A = -25 \quad A = 5$$

$$x_p = 5 + 8t$$

$\boxed{x(t) = c_1 e^{-5t} + c_2 e^{t} + 8t + 5}$ is the G. solution of (1)

But $4y = 3x - x'(t) + 1$

Then $4y(t) = 3c_1 e^{-5t} + 3c_2 e^{t} + 24t + 15 - \left[-c_1 5 e^{-5t} + c_2 e^{t} + 8\right] + 1$

$= 3c_1 e^{-5t} + 3c_2 e^{t} + 24t + 15 + c_1 5 e^{-5t} - c_2 e^{t} - 8 + 1$

$$4y(t) = 8c_1 e^{-5t} + 2c_2 e^{t} + 24t + 8$$

$$\boxed{y(t) = 2c_1 e^{-5t} + \frac{1}{2} c_2 e^{t} + 6t + 2}$$