Solution of the Final Examination   Math 244   Semester II - 1445

Question 1 :

Question	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)
Solution	d)	b)	d)	b)	a)	b)	b)	b)	d)	c)

Question 2 : [Marks: 2+2+3]

a) $P(A) = I_3$. $= A^3 - 3A^2 - 4A + 13I_3$

b) $\begin{vmatrix} 1 & 2 & 2 \\ x+1 & 2x+1 & 2x+2 \\ x+1 & x+1 & 2x+1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x+1 & -1 & 1 \\ x+1 & -x-1 & x \end{vmatrix} = 1.$ (There is more than one way to do it)

c) If $aX_1 + bX_2 = 0$, then $A(aX_1 + bX_2) = aB + bC = 0$, then $a = b = 0$, since B, C are linearly independent.
B and C are in the (image space) of A and linearly independent. Then $\text{rank}(A) \geq 2$.
($\equiv$ Column space)

Question 3 : [Marks: 2+3+3]

a) The RREF of A is the matrix $\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

b) $B = \{(1, -1, 0, 1), (-1, 1, 1, 0)\}$ i a basis of the column space $\text{Col}(A)$
and $C = \{(1, 1, -1, 0), (-2, -3, 0, 1)\}$ is a basis of the null space $N(A)$.

c) $\text{rank}(A) = 2$ and $\text{Nullity}(A) = 2$. Since Nullity(A) = # of columns of A − rank(A).

Question 4 : [Marks: 3+2+3]

a) $\langle 1, x^2 \rangle = 2$. Then $\{1, x, x^2\}$ is not orthogonal.
$(\frac{1}{\sqrt{3}}, \frac{x}{\sqrt{2}}, \frac{1}{\sqrt{6}}(3x^2 - 2))$ is the orthonormal basis.

b) If $au + bv + cw = 0$, then taking the inner product with u, v and w respectively, we get $a = 0$, $b = 0$ and $c = 0$.

c) The matrix of T with respect to the basis $B = \{v_1 = (2, 2, 1), v_2 = (2, 1, 0), v_3 = (1, 0, 0)\}$ and the standard basis of $\mathbb{R}^2$ is $A = \begin{pmatrix} 3 & 6 & 4 \\ -1 & 2 & 3 \end{pmatrix}$. Then $\text{rank}(T) = 2$ and $\text{nullity}(T) = 1$.
The set of solutions of the system $AX = 0$ is $\{(10, -13, 12)t : t \in \mathbb{R}\}$. Then

$$\ker(T) = \{(10v_1 - 13v_2 + 12v_3)t : t \in \mathbb{R}\}.$$

Another way: $\{(3,-1),(6,2)\} \subseteq \operatorname{Im} T \subseteq \mathbb{R}^2$ is L.I. $\Rightarrow$ rank $T = 2$.

$\Rightarrow \dim \ker T = \dim \mathbb{R}^3 - \operatorname{rank} T = 3-2 = 1$.

$\Rightarrow$ Any $0 \neq u \in \ker T$ spans $\ker T$. Let $u = a v_1 + b v_2 + c v_3$. Then $T(u) = 0$

$\Leftrightarrow a T(v_1) + b T(v_2) + c T(v_3) = 0$

$a(3,-1) + b(6,2) + c(4,3) = (0,0)$

$$\Leftrightarrow \begin{matrix} 3a + 6b + 4c = 0 \\ -a + 2b + 3c = 0 \end{matrix} \Leftrightarrow \begin{matrix} a - 2b - 3c = 0 \\ 3a + 6b + 4c = 0 \end{matrix} \Leftrightarrow \begin{matrix} a - 2b - 3c = 0 & \dots & (1) \\ 12b + 13c = 0 & \dots & (2) \end{matrix}$$

Put $C = -12$ in $(2) \Rightarrow b = 13 \xRightarrow{\text{in}(1)} a = 2b + 3c = 26 - 36 = -10$

$\therefore \; u = -10(2,2,1) + 13(2,1,0) + (-12)(1,0,0) = (-6,-7,-10)$

$\Rightarrow \ker T = \{ t(-6,-7,-10) : t \in \mathbb{R} \}$

3rd way: By finding the standard matrix for T:

$T(1,0,0) = (4,3)$

$T(0,1,0) = T(2,1,0) - 2T(1,0,0) = (6,2) - (8,6) = (-2,-4)$

$T(0,0,1) = T(2,2,1) - 2T(1,0,0) - 2T(0,1,0) = (3,-1) - 2(4,3) - 2(-2,-4)$
$= (-1,1)$

$$\Rightarrow A = \begin{bmatrix} 4 & -2 & -1 \\ 3 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow{A_{21}^{(-1)}} \begin{bmatrix} 1 & 2 & -2 \\ 3 & -4 & 1 \end{bmatrix} \xrightarrow{A_{12}^{(-3)}} \begin{bmatrix} 1 & 2 & -2 \\ 0 & -10 & 7 \end{bmatrix} \xrightarrow{M_2^{-\frac{1}{10}}} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -\frac{7}{10} \end{bmatrix}$$

$$\xrightarrow{A_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & -\frac{3}{5} \\ 0 & 1 & -\frac{7}{10} \end{bmatrix} \Rightarrow \ker T = \operatorname{Span}\{(\tfrac{3}{5}, \tfrac{7}{10}, 1)\}$$

$\operatorname{rank} T = 2$

Question 5 : [Marks: 3+4]

a) $Y = C^{-1}X$: $AX = \lambda X$. $B = C^{-1}AC \Rightarrow BC^{-1} = C^{-1}A \Rightarrow BC^{-1}X = C^{-1}AX = C^{-1}\lambda X = \lambda C^{-1}X$.

b)

$$\begin{aligned}
q_A(\lambda) &= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 2-\lambda & 2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (2+\lambda)\begin{vmatrix} -1 & 2 & 3 \\ 0 & 2-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} \\
&= (2+\lambda)\begin{vmatrix} -1 & 2 & 3 \\ 0 & 2-\lambda & 2 \\ 0 & 4 & 4-\lambda \end{vmatrix} = -(2+\lambda)\begin{vmatrix} -\lambda & 2 \\ \lambda & 4-\lambda \end{vmatrix} \\
&= -\lambda(2+\lambda)(\lambda-6).
\end{aligned}$$

For $\lambda = 0$, $X_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. For $\lambda = -2$, $X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. For $\lambda = 6$, $X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$P = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$.

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 & -3 \\ -2 & \lambda-2 & -2 \\ -3 & -2 & \lambda-1 \end{vmatrix} \overset{A_{31}^{(-1)}}{=} \begin{vmatrix} \lambda+2 & 0 & -\lambda-2 \\ -2 & \lambda-2 & -2 \\ -3 & -2 & \lambda-1 \end{vmatrix}$$

$$= (\lambda+2) \begin{vmatrix} 1 & 0 & -1 \\ -2 & \lambda-2 & -2 \\ -3 & -2 & \lambda-1 \end{vmatrix} = (\lambda+2) \begin{vmatrix} 1 & 0 & 0 \\ -2 & \lambda-2 & -4 \\ -3 & -2 & \lambda-4 \end{vmatrix}$$

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$$= (\lambda+2)\left[(\lambda-2)(\lambda-4) - 8\right] = (\lambda+2)\left[\lambda^2 - 6\lambda + 8 - 8\right]$$

$$= (\lambda+2)\lambda(\lambda-6)$$