[DRAFT]

King Saud University
College of Sciences
Department of Mathematics
Semester 451 / Final Exam / MATH-244 (Linear Algebra)

Max. Marks: 40 **Time: 3 hours**

Name: **ID:** **Section:** **Signature:**

Note: ***Attempt all the five questions. Scientific calculators are not allowed***!

Question 1 [Marks:10 $\times$1]**:**

Choose the correct answer:

(i) If the matrix $A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -4 \\ 1 & -\delta & 0 \end{bmatrix}$ satisfies the condition $-A = A^t$ (here, A^t denotes the transpose of A), then δ is equal to:

(a) 0 (b) -4 (c) -2 (d) 4.

(ii) If A and B are 3×3 matrices with $|A| = -1$ and $|3A^2BA^{-1}| = -54$, then $|B|$ is equal to:

(a) 18 (b) -18 (c) -2 (d) 2.

(iii) If $|3A| = -2$, then the reduced row echelon form of A must be equal to:

(a) $3\mathbf{I}$ (b) $2\mathbf{I}$ (c) $\frac{1}{3}\mathbf{I}$ (d) $\mathbf{I}$.

(iv) Let F denote the set of all solutions of the linear system $AX = O$, where the matrix of A is invertible and $X \in \mathbb{R}^3$. Then, F is equal to:

(a) $\mathbb{R}^3$ (b) {} (c) {(0,0.0)} (d) $\mathbb{R}^3 - \{(0,0,0)\}$.

(v) If the vectors (1,2,1), (2,5,3) and $(-1,-4,h)$ are linearly dependent in $\mathbb{R}^3$, then h is equal to:

(a) 5 (b) -3 (c) -5 (d) 3

(vi) If $W = span\,\{(1,1,1),(-2,-2,-2)\}$, then $\dim(W)$ is equal to:

(a) 2 (b) -1 (c) 0 (d) 1.

(vii) If $B = \{(-2,4),(3,-5)\}$ is an ordered basis of the vector space $\mathbb{R}^2$, then the coordinate vector $[(1,3)]_B$ is equal to:

(a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$.

(viii) If the inner product on the vector space P_2 of polynomials with degree ≤ 2 is defined by $\langle p,q \rangle = a_0b_0 + a_1b_1 + a_2b_2$ for all $p = a_0 + a_1x + a_2x^2, q = b_0 + b_1x + b_2x^2 \in P_2$, then the distance between the polynomials $2 - 3x^2$ and $1 - x + x^2$ is equal to:

(a) 1 (b) $3\sqrt{2}$ (c) 0 (d) $2\sqrt{3}$.

(ix) If $T: P_2 \to P_3$ is the linear transformation defined by $T(p(x)) = xp(x)$, then which of the following polynomial is in $Im(T)$?

(a) $3 - x^2$ (b) $1 + x^3$ (c) $3x - x^2$ (d) $3x - x^4$.

(x) If $A = \begin{bmatrix} 0 & -1 \\ -4 & 0 \end{bmatrix}$, then the matrix A is:

(a) diagonalizable (b) symmetric (c) not diagonalizable (d) not invertible.

Question 2 [Marks: 2+3+1]**: :** Let A be a matrix with $RREF(A) = \begin{bmatrix} 1 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then:

(a) Find $rank(A)$ and $nullity(A)$.
(b) Find a basis for the row space $row(A)$ and the null space $N(A)$.
(c) Show whether or not the linear system $AX = b$ has a solution for all $b \in \mathbb{R}^5$.

Question 3 [Marks: 4+5]**:**

(a) Let $\boldsymbol{A}$ = {(1,0,3), (1,1,0), (2,2, -3)} and $\boldsymbol{B}$ = {(1,0,0), (0,1,0), (0,0,1)} be given ordered bases for the vector space $\mathbf{R}^3$. Then construct the change of basis matrix ${}_{\boldsymbol{A}}\mathbf{P}_{\boldsymbol{B}}$ and then use it to find the coordinate vector $[v]_A$ of $v = (3.-2.1)$.
(b) Let $E = \{u_1 = (1,1,1), u_2 = (-1,1,0), u_3 = (1,2,1)\}$. Find an orthonormal basis for the Euclidean space $\mathbb{R}^3$ by applying the Gram-Schmidt algorithm on E.

Question 4: [Marks: 2+3+2]

Let T: $\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by:

$$T(1,0,3) = (2,-1,3), T(0,1,0) = (1,0,2), \text{ and } T(0,0,2) = (4,0,-2).$$

(a) Find the standard matrix for the transformation T.
(b) Find $\dim(\ker(T))$ and $\dim(\operatorname{Im}(T))$.
(c) Find $T(1,2,3)$ by using the standard matrix of T.

Question 5: [Marks: 3+3+2]

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ -3 & 0 & -2 \end{bmatrix}$. Then:

(a) Show that the matrix A is diagonalizable.
(b) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.
(c) Compute the matrix A^{2024}.

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