[Solution Key]

King Saud University
College of Sciences
Department of Mathematics
Semester 451 / Final Exam / MATH-244 (Linear Algebra)

Max. Marks: 40 **Time: 3 hours**

Name: **ID:** **Section:** **Signature:**

Note: ***Attempt all the five questions. Scientific calculators are not allowed***!

Question 1 [Marks:10 ×1]**:**

Choose the correct answer:

(i) If the matrix $A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -4 \\ 1 & -\delta & 0 \end{bmatrix}$ satisfies the condition $-A = A^t$ (here, A^t denotes the transpose of A), then δ is equal to:

(a) 0 (b) ✓ -4 (c) -2 (d) 4.

(ii) If A and B are 3×3 matrices with $|A| = -1$ and $|3A^2BA^{-1}| = -54$, then $|B|$ is equal to:

(a) 18 (b) -18 (c) -2 (d) ✓ 2.

(iii) If $|3A| = -2$, then the reduced row echelon form of A must be equal to:

(a) $3\mathbf{I}$ (b) $2\mathbf{I}$ (c) $\frac{1}{3}\mathbf{I}$ (d) ✓ $\mathbf{I}$.

(iv) Let F denote the set of all solutions of the linear system $AX = O$, where the matrix of A is invertible and $X \in \mathbb{R}^3$. Then, F is equal to:

(a) $\mathbb{R}^3$ (b) {} (c) ✓ {(0,0.0)} (d) $\mathbb{R}^3 - \{(0,0,0)\}$.

(v) If the vectors (1,2,1), (2,5,3) and $(-1,-4,h)$ are linearly dependent in $\mathbb{R}^3$, then h is equal to:

(a) 5 (b) ✓ -3 (c) -5 (d) 3

(vi) If $W = span\{(1,1,1),(-2,-2,-2)\}$, then $\dim(W)$ is equal to:

(a) 2 (b) -1 (c) 0 (d) ✓ 1.

(vii) If $B = \{(-2,4),(3,-5)\}$ is an ordered basis of the vector space $\mathbb{R}^2$, then the coordinate vector $[(1,3)]_B$ is equal to:

(a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (c) ✓ $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$.

(viii) If the inner product on the vector space P_2 of polynomials with degree ≤ 2 is defined by $\langle p,q \rangle = a_0b_0 + a_1b_1 + a_2b_2$ for all $p = a_0 + a_1x + a_2x^2, q = b_0 + b_1x + b_2x^2 \in P_2$, then the distance between the polynomials $2 - 3x^2$ and $1 - x + x^2$ is equal to:

(a) 1 (b) ✓ $3\sqrt{2}$ (c) 0 (d) $2\sqrt{3}$.

(ix) If $T: P_2 \to P_3$ is the linear transformation defined by $T(p(x)) = xp(x)$, then which of the following polynomial is in $Im(T)$?

(a) $3 - x^2$ (b) $1 + x^3$ (c) ✓ $3x - x^2$ (d) $3x - x^4$.

(x) If $A = \begin{bmatrix} 0 & -1 \\ -4 & 0 \end{bmatrix}$, then the matrix A is:

(a) ✓ diagonalizable (b) symmetric (c) not diagonalizable (d) not invertible.

Question 2 [Marks: 2+3+1]**: :** Let A be a matrix with $RREF(A) = \begin{bmatrix} 1 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then:

(a) Find $rank(A)$ and $nullity(A)$.
(b) Find a basis for the row space $row(A)$ and the null space $N(A)$.
(c) Show whether or not the linear system $AX = b$ has a solution for all $b \in \mathbb{R}^5$.

Solution: (a) $rank(A) =$ the number of nonzero rows in $RREF(A) = 4$.
Since $rank(A)+ nullity(A) = the\ number\ of\ columns\ in\ A$, we have $nullity(A) = 6 - 4 = 2$..

(b) A basis for $row(A)$ consists of the nonzero rows in $RREF(A)$:
$\{(1,1,0,-3,0,0), (0,0,1,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1)\}$..
Since $dimN(A) = 2$ and since $(3,0,-1,1,0,0), (0,3,-1,1,0,0) \in N(A)$ are linearly independent, $\{(3,0,-1,1,0,0), (0,3,-1,1,0,0)\}$ is a basis for $N(A)$.
[$\{(3,0,-1,1,0,0), (-1,1,0,0,0,0)\}$ is another basis for $N(A)$.]

(c) Since $dim\ col(A) = rank(A) = 4 < 5 = dim\mathbb{R}^5$, the columns of A do not span $\mathbb{R}^5$. Hence, there exists at least on $b \in \mathbb{R}^5$ for which $AX = b$ has no solution.

Question 3 [Marks: 4+5]**:**

(a) Let $\boldsymbol{A}$ = {(1,0,3), (1,1,0), (2,2, -3)} and $\boldsymbol{B}$ = {(1,0,0), (0,1,0), (0,0,1)} be given ordered bases for the vector space $\mathbf{R}^3$. Then construct the change of basis matrix $_{\boldsymbol{A}}\mathbf{P}_{\boldsymbol{B}}$ and then use it to find the coordinate vector $[v]_A$ of $v = (3.-2.1)$.

(b) Let $E = \{u_1 = (1,1,1), u_2 = (-1,1,0), u_3 = (1,2,1)\}$. Find an orthonormal basis for the Euclidean space $\mathbb{R}^3$ by applying the Gram-Schmidt algorithm on E.

Solution: (a) $(1,0,0) = \alpha(1,0,3) + \beta(1,1,0) + \gamma(2,2,-3) \Rightarrow \alpha = 1 = \gamma, \beta = -2$ so that $[(1,0,0)]_A = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Similarly,

$[(0,1,0)]_A = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ and $[(0,0,1)]_A = \begin{bmatrix} 0 \\ 2/3 \\ -1/3 \end{bmatrix}$. Hence, $_A\mathbf{P}_B = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & \frac{2}{3} \\ 1 & -1 & -\frac{1}{3} \end{bmatrix}$,

Next, $[v]_A = {}_A\mathbf{P}_B\,[v]_B = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & \frac{2}{3} \\ 1 & -1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -\frac{34}{3} \\ \frac{14}{3} \end{bmatrix}$.

(b) $v_1 = u_1 = (1,1,1);\ v_2 = u_2 - \frac{<u_2,u_1>}{||u_1||^2} u_1 = u_2 - \frac{0}{||u_1||^2} u_1 = (-1,1,0)$ and $v_3 = u_3 - \frac{<u_3,u_1>}{||u_1||^2} u_1 - \frac{<u_3,u_2>}{||u_2||^2} u_2 = (\frac{1}{6}, \frac{1}{6}, \frac{-1}{3})$.

Hence, $w_1 = \frac{1}{||v_1||} v_1 = \frac{1}{\sqrt{3}}(1,1,1), w_2 = \frac{1}{||v_2||} v_2 = \frac{1}{\sqrt{2}}(-1,1,0), w_3 = \frac{1}{||v_3||} v_3 = \sqrt{6}\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$.

Question 4: [Marks: 2+3+2]

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by:

$$T(1,0,3) = (2,-1,3), T(0,1,0) = (1,0,2), \text{ and } T(0,0,2) = (4,0,-2).$$

(a) Find the standard matrix for the transformation T.
(b) Find $\dim(\ker(T))$ and $\dim(\mathrm{Im}(T))$.
(c) Find $T(1,2,3)$ by using the standard matrix of T.

Solution: (a) $(4,0,-2) = T(0,0,2) = 2T(0,0,1) \Rightarrow T(0,0,1) = (2,0,-1)$. and so
$(2,-1,3) = T\big((1,0,0) + 3(0,0,1)\big) = T(1,0,0) + 3T(0,0,1) = T(1,0,0) + 3(2,0,-1)$
gives $T(1,0,0) = (-4,-1,6)$.

Hence, the standard matrix for the transformation T is $\begin{bmatrix} -4 & 1 & 2 \\ -1 & 0 & 0 \\ 6 & 2 & -1 \end{bmatrix}$.

(b) Since $\begin{vmatrix} -4 & 1 & 2 \\ -1 & 0 & 0 \\ 6 & 2 & -1 \end{vmatrix} = -5 \neq 0$, we have $\dim(\mathrm{Im}(T)) = \mathrm{rank}(T) = 3$.

Hence, $\dim(\ker(T)) = \dim(\mathbb{R}^3) - \mathrm{rank}(T) = 3 - 3 = 0$.

(c) $T(1,2,3) = \begin{bmatrix} -4 & 1 & 2 \\ -1 & 0 & 0 \\ 6 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$..

Question 5: [Marks: 3+3+2]

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ -3 & 0 & -2 \end{bmatrix}$. Then:

(a) Show that the matrix A is diagonalizable.

(b) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

(c) Compute the matrix A^{2024} .

Solution: (a) $0 = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ -3 & 0 & -2-\lambda \end{vmatrix} = -(\lambda+1)(\lambda-1)^2 \Rightarrow \lambda = -1, 1, 1$ are the eigenvalues of A.

The eigenspace corresponding to the eigenvalue $\lambda = -1$: $E_{-1} = $ < (-1/3, 0, 1)> and

the eigenspace corresponding to the eigenvalue $\lambda = 1$: $E_1 = $ <{ (-1, 0, 1), (0,1,0)}>.

So, the algebraic multiplicity of each eigenvalue of A equals to its geometric multiplicity.

Hence, the matrix A is diagonalizable.

(b) From Part (a), we get an invertible matrix $P = \begin{bmatrix} -1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ with $P^{-1} = \begin{bmatrix} -\frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{3}{2} & 0 & \frac{3}{2} \end{bmatrix}$ so that

$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D\ (say)$; which is a diagonal matrix.

(c) From Part (b), we get $A^{2024} = (PDP^{-1})^{2024} = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{2024} P^{-1} = PIP^{-1} = \mathrm{I}$.

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