College of Sciences
Department of Statistics & OR

كلية العلوم
قسم الإحصاء وبحوث العمليات

جامعة الملك سعود
King Saud University

**Final Exam, Academic Year 1442 H, Second semester**

| Exam Information | | معلومات الامتحان | |
|---|---|---|---|
| Course name | Actuarial probability | احتمال اكتواري | اسم المقرر |
| Course Code | STAT 216 | 216 احص | رمز المقرر |
| Exam Date | 2021-05-05 | 1442-09-23 | تاريخ الامتحان |
| Exam Time | 01:30PM | 01:30 م | وقت الامتحان |
| Exam Duration | 3 hours | 03 ساعات | مدة الامتحان |
| Classroom No. | | | رقم قاعة الاختبار |
| Instructor Name | | | اسم استاذ المقرر |

| Student Information | | معلومات الطالب | |
|---|---|---|---|
| Student's Name | | | اسم الطالب |
| ID number | | | الرقم الجامعي |
| Section No. | 66116 | 66116 | رقم الشعبة |
| Serial Number | | | الرقم التسلسلي |

**General Instructions:**

- Your Exam consists of 3 pages (except this paper)
- Keep your mobile and smart watch out of the classroom .

**تعليمات عامة:**

- عدد صفحات الامتحان 3 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف والساعات الذكية خارج قاعة الامتحان.

هذا الجزء خاص بأستاذ المادة — *This section is ONLY for instructor*

| # | Course Learning Outcomes (CLOs) | Related Question(s) | Points | Final Score |
|---|---|---|---|---|
| 1 | General Probability | Q 7 | | |
| 2 | Univariate random variables | Q 4-5 | | ___ / 20 |
| 3 | Multivariate random variables | Q 1-2-3-6-8 | | |

**Exercise 1:** Let $(X,Y)$ a r.v. in $\mathbb{R}^2$ admitting the density $f_{(X,Y)}$ given by

$$f_{(X,Y)}(x,y) = \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi}, \quad x,y \in \mathbb{R}.$$

1) Are the random variables $X$ and $Y$ independent?
2) Find the distribution of the r.v. $Z = \dfrac{X+Y}{2}$.

**Exercise 2:** *We recall the Gamma function is given by* $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$, $t > 0$ *and we admit that*

$$\int_0^\infty \frac{x^{t-1}}{\Gamma(t)} e^{-zx}\, dx = \frac{1}{z^t}, \quad \forall z = x + iy \in \mathbb{C},\ x > 0.$$

Let $X$ and $Y$ two **independent** r.v. having respectively a $Gamma(\lambda, s)$ and $Gamma(\lambda, t)$ distribution ($\lambda, s, t > 0$), i.e. their densities are given by

$$f_X(x) = \frac{\lambda^s}{\Gamma(s)} \begin{cases} x^{s-1} e^{-\lambda x}, & \text{if } x > 0, \\ 0, & otherwise \end{cases} \quad \text{and} \quad f_Y(x) = \frac{\lambda^t}{\Gamma(t)} \begin{cases} x^{t-1} e^{-\lambda x}, & \text{if } x > 0, \\ 0, & otherwise. \end{cases}$$

1) Show that the moment generating function of $X$, $M_X(u) = \mathbf{E}[e^{uX}] = \int_{\mathbb{R}} e^{ux} f_X(x)\, dx$, has the form:

$$M_X(u) = \left(\frac{\lambda}{\lambda - u}\right)^s, \quad u < \lambda.$$

2) Deduce the moment generating function $M_Y$ of $Y$.
3) Express the moment generating function $M_{\lambda X}$ of the r.v. $\lambda X$.
4) What is the distribution of $\lambda X$?
5) Express the moment generating function of the r.v. $Z = \lambda(X+Y)$, $M_Z(u) = \mathbf{E}[e^{\lambda u (X+Y)}]$.
6) What is the distribution of $Z$?

**Exercise 3:** Let $(X,Y)$ a couple of random variables with density function given by:

$$f(x,y) = 12x^2y^3\, 1_{(x,y)\in(0,1)^2}.$$

Compute $\mathbb{P}(X > \frac{1}{2}, Y < \frac{1}{2},)$.

**Exercise 4:** We throw two dices: one black, one red, both are numerated from 1 to 6. We denote $Z$ the random variable that gives the sum observed on the two dices.

1) Calculate $\mathbb{P}(Z = 3)$.

2) We make the following game: we continue to throw the two dices and we win as soon as we obtain a sum equal 5. We have then generated a sequence of i.i.d. random variables $X_1, X_2, \ldots, X_n, \ldots$, given by

$$X_n = \begin{cases} 1 & \textit{if the } n^{\text{th}} \textit{ throw gives a sum equal to } 3 \\ 0 & \textit{otherwise}. \end{cases}$$

We also denote $T$ the time at which the game stops, that means that the event $(T = n)$ is given by

$$(T = n) = (X_1 = 0, X_2 = 0, \cdots, X_n = 1), \quad n \geq 1.$$

2.a) Calculate $P(X_n = 0)$ and $P(X_n = 1)$.
2.b) What is the distribution of $X_n$ ?
2.c) What is the distribution of $T$ ?
2.d) In average, how long should we wait before winning ?

**Exercise 5:** Let $X$ a discrete random variable such that

$$\mathbb{P}(X = k) = ck^2, \quad k = 1,2,3,4,5.$$

1) Find the value of $c$.
2) Calculate $\mathbb{E}[X]$.

**Exercise 6:** Let $X$ a r.v. with uniform distribution over $[0,1]$:

$$f_X(u) = 1 \text{ if } u \in [0,1], \qquad f_X(u) = 0 \text{ otherwise.}$$

1) Express the cumulative distribution function $F_X(x) = \mathbb{P}(X \leq x)$.
2) Calculate $P(0 \leq X \leq 1)$, $P(X \leq 0)$ and $P(X \geq \frac{1}{2})$.

**Exercise 7:** An insurance company looks at its auto insurance customers and finds that:

(a) all insure at least one car;
(b) 75% insure more than one car;
(c) 35% insure a sports car;
(d) 10% insure more than one car, including a sports car.

Find the probability that a customer selected at random insures exactly one car and it is not a sports car.

**Exercise 8:** In an insurance company, a model describes the time until a loss $X$ occurs. The size of the loss is denoted $Y$. We know that $X$ has pdf

$$f_X(x) = \frac{1}{x^2}, \quad x > 1,$$

and that the conditional distribution of $Y$ given $X = x$ has pdf

$$f_{Y|X=x}(y) = \frac{1}{x}, \text{ for } \quad x < y < 2x.$$

Find the distribution of $Y$.