Solution Key:

King Saud University
College of Sciences
Department of Mathematics
Math-244 (Linear Algebra); Final Exam; Semester 442

Max. Marks: 40 **Time: 3 hours**

Question 1 [Marks: 5+5]**:**

I. Choose the correct answer:

(i) Let B and C be ordered bases of a vector space V with transition matrix ${}_C\boldsymbol{P}_{\boldsymbol{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. If the coordinate vector $[\boldsymbol{v}]_C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ then the coordinate vector $[\boldsymbol{v}]_B$ is:

(a) $(1,3,6)$ (b) $(6,3,1)$ (c) ✓ $(1,1,1)$ (d) $(1,1,2)$.

(ii) The dimension of the column space $col(A^t)$ of $A = \begin{bmatrix} 2 & 3 & 1 & -1 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is:

(a) 1 (b) ✓ 2 (c) 4 (d) 5.

(iii) If $U = \begin{bmatrix} -1 & 3 \\ y & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 5 & 2y \\ -4 & 1 \end{bmatrix}$ are two orthogonal matrices with respect to the inner product $< A, B > = trace\,(AB^t)$ on the vector space M_2 of 2x2 real matrices, then:

(a) ✓ $y = 2$ (b) $y = -2$ (c) $y = 0$ (d) $y = 1$.

(iv) If the inner product on the vector space $\boldsymbol{P_2}$ of polynomials with degree ≤ 2 is defined by $\langle p,q \rangle = aa_1 + 2bb_1 + cc_1$, $\forall\, p = a + bx + cx^2$, $q = a_1 + b_1x + c_1x^2 \in \boldsymbol{P_2}$ and θ is the angle between the polynomials $1 + x - x^2$ and $2 + x - 2x^2$, then:

(a) $\cos\theta = \frac{5}{3\sqrt{3}}$ (b) $\cos\theta = \frac{2}{\sqrt{3}}$ (c) ✓ $\cos\theta = \frac{3}{\sqrt{10}}$ (d) $\cos\theta = 1$.

(v) If $S = \{v_1 = (2,1), v_2 = (1,0)\}$ is a basis for Euclidean space $\mathbb{R}^2$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation defined by $T(v_1) = (1,5)$ and $T(v_2) = (0,3)$, then $T(4,6)$ is equal to:

(a) ✓ $(6,6)$ (b) $(-8,-22)$ (c) $(-10,-8)$ (d) $(4,23)$.

II. Determine whether the following statements are true or false; justify your answer.

(i) If $\{(-3r + 4s, r - s, r, s): r, s \in \mathbb{R}\}$ is the solution space of homogeneous system $AX = 0$, then $nullity(A) = 2$.

True: {(-3,1,1,0), (4,-1,0,1)} is a basis for the null space.

(ii) For any $m \times n$ matrix A, $dim(\boldsymbol{N}(A^t)) + dim(col(\mathrm{A})) = m$.

True: $dim(row(A^t)) = dim(col(\mathrm{A})) \Rightarrow dim(\boldsymbol{N}(A^t)) + dim(row(A^t)) = nullity(A^t) + rank(A^t) = m$.

(iii) The transformation $T: \mathbb{R} \to \mathbb{R}$ given by $T(r) = |r|$ is linear.

False: for example, $|-1 + 1| = 0 \neq 2 = |-1|+|1|$. (Or, $|(-1)1| = 1 \neq -1 = -1|1|$.

(iv) Eigenvalues of any matrix are same as the eigenvalues of its reduced row echelon form.

False: for example, 2 is an eigenvalue of $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ but neither of the two eigenvalues of its RREF *is* 2.

(v) If the characteristic polynomial of a matrix A is $q_A(\lambda) = \lambda^2 - 2$, then A is diagonalizable.

True: the quadratic characteristic polynomial gives two different eigenvalues of 2x2 matrix A.

Question 2 [Marks: 2.5+1+2.5]**:** Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$. Then:

(i) Find a basis for $col(A)$.

Solution: $= \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (REF). So, {(1,-1,0,1), (0,1,2,1), (3,-3,1,4)} is a basis for col(A).

(ii) Find $dim(row(A))$.

Solution: $dim(row(A)) = dim(col(A)) = 3$ as is clear from the above Part (i).

(iii) Find a basis for the null space $\boldsymbol{N}(A)$.

Solution: The REF of A obtained in Part (i) gives the basis {(2,-3,1,0,0),(-1,0,0,1,0)} for $N(A)$.

Question 3 [Marks: 3+3]**:** Let $E = \{v_1 = (1,1,-4,-3),\ v_2 = (2,0,2,-2),\ v_3 = (2,-1,3,2)\}$. Then:

(i) Find a basis B for the vector space $span(E)$ such that $B \subseteq E$. If $E - B \neq \phi$, then express each element of $E - B$ as linear combination of the basic vectors.

Solution: The set E is linearly independent and so B ; = E is a basis for $span(E)$; this completes the solution of Part (i).

(ii) Use the basis B (as in Part (i)) to find a basis $\boldsymbol{C}$ for the Euclidean space $\mathbb{R}^4$.

Solution: It is easily seen that $\{v_1, v_{2,}, v_3, (1,0,0,0,))\}$ being linearly independent is a basis for the space $\mathbb{R}^4$.

Question 4: [Marks: 2+4]

a) Let $\{u, v, w\}$ be an orthogonal set of vectors in an inner product space. Then show that:

$$\|u\|^2 + \|v\|^2 + \|w\|^2 = \|u + v + w\|^2.$$

Solution: $\|u+v+w\|^2 = < u+v+w\,, u+v+w > = \|u\|^2 + 2(< u,v > + < u,w > + < v,u >) + \|v\|^2 + \|w\|^2$. Hence, by the given orthogonality of $\{u, v, w\}$, $\|u + v + w\|^2 = \|u\|^2 + \|v\|^2 + \|w\|^2$.

b) Let $A = \{\boldsymbol{u_1} = (1,1,1), \boldsymbol{u_2} = (0,1,-1), \boldsymbol{u_3} = (3,-2,2)\}$. Use the Gram-Schmidt algorithm to obtain an orthonormal set B of vectors such that $span(B) = span(A)$.

Solution: Put $v_1 := \boldsymbol{u_1} = (1,1,1)$, $v_2 := \boldsymbol{u_2} - \frac{< u_2, v_1 >}{\|v_1\|^2} v_1 = \boldsymbol{u_2} = (0,1,-1)$, $v_2 := \boldsymbol{u_3} - \frac{< u_3, v_1 >}{\|v_1\|^2} v_1 - \frac{< u_3, v_2 >}{\|v_2\|^2} v_2 = (2,-1,-1)$.

Then, $B := \{\boldsymbol{w_1} := \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{3}}(1,1,1), \boldsymbol{w_2} := \frac{1}{\|v_2\|} v_2 = \frac{1}{\sqrt{2}}(0,1,-1), \boldsymbol{w_3} := \frac{1}{\|v_3\|} v_3 = \frac{1}{\sqrt{6}}(2,-1,-1)\}$ is as required.

Question 5: [Marks: (2+1.5+2.5) + (2.5+1+2.5)]

a) Let the linear transformation T: $M_2 \to \mathbb{R}^2$ be defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b)$, $\forall\ a,b,c,d \in \mathbb{R}$, Then find:

(i) A basis for $ker(T)$.

Solution: Clearly, $ker(T) = \left\{\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} : c, d \in \mathbb{R}\right\} = span(\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\})$; hence, $\left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ being linearly independent in M_2 is a basis for $ker(T)$.

(ii) $rank(T)$.

Solution: $rank(T) = dim(M_2) - nullity(T) = 4 - 2 = 2$ from the solution of Part (i).

(iii) The standard matrix $[T]_B^C$, where B and C are the standard bases of M_2 and $\mathbb{R}^2$, respectively.

Solution: $[T]_B^C = \left[\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_C \left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_C \left[T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_C \left[T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_C\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$..

b) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ \boldsymbol{x} & 2 & 0 \\ \boldsymbol{y} & \boldsymbol{z} & -3 \end{bmatrix}$. Then:

(i) Find the values of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\mathbf{z}$ such that $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -3$ are the eigenvalues of A with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, respectively.

Solution: $\begin{bmatrix} 1 & 0 & 0 \\ \boldsymbol{x} & 2 & 0 \\ \boldsymbol{y} & \boldsymbol{z} & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ x+2 \\ y+z-3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow x+2=1, y+z-3=1$; and $\begin{bmatrix} 1 & 0 & 0 \\ \boldsymbol{x} & 2 & 0 \\ \boldsymbol{y} & \boldsymbol{z} & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \mathbf{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow z-3=2.$

Hence, $x=-1,\ y=-1,\ z=5.$

(ii) Use the values of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ (as in Part (i)) to show that the matrix A is diagonalizable.

Solution: Since A is a 3×3 matrix having 3 different eigenvalues, it is diagonalizable.

(iii) Find A^5.

Solution: Since the matrix A is diagonalizable having eigenvalues 1, 2, -3 with corresponding eigen vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, respectively, we get $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ with $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ satisfying $A = PD\,P^{-1}$. Hence, $A^5 = P\,D^5\,P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-31} & 32 & 0 \\ \mathbf{-31} & \mathbf{275} & -243 \end{bmatrix}$.

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