KING SAUD UNIVERSITY,DEPARTMENT OF MATHEMATICS
MATH 204. TIME: 3H, FULL MARKS: 40, FINAL EXAM
(T2-2023)

Question 1. **[5,4] a)** Solve the initial value problem

$$\begin{cases} y'' + 5y' + 6y = 2e^{-3x} + 4e^{-2x} \\ y(0) = 0, \ y'(0) = 0. \end{cases}$$

b) Obtain the solution of the differential equation

$$\left(\frac{1}{x} + 6xy + 4xye^{x^2}\right) dx + \left(\frac{1}{y} + 3x^2 + 2e^{x^2}\right) dy = 0, \ x > 0, \ y > 0.$$

Question 2. **[4,5] a)** If

$$y'' - 6y' + 9y = e^{3x}, \ y(0) = 1, \ y(1) = e^3,$$

then find the value of $y(2)$.

b) Use power series method near the ordinary point $x_0 = 0$, to find the first five terms of the solution for the differential equation

$$y'' + xy' + x^2y = 0.$$

Question 3. **[4,3,5] a)** Find only the form of y_p for the differential equation

$$y'' - 4y' + 4y = xe^{2x} + x^3e^{2x} + e^{2x}\sin(2x).$$

b) Determine the geneal solution of the homogeneous differential equation having the characteristic equation

$$m^2(m^4 - 16)(m - 3)^2 = 0.$$

c) Solve the the differential equation

$$xy'' - y' - \frac{3}{x}y = x\ln x, \ x > 0.$$

Question 4. **[5,5] a)** Find the Fourier sine series for the function $f(x) = 1+x$, $0 \le x \le 1$, and deduce that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$.

b) Consider the function: $f(x) = \begin{cases} 0, & x \le -1 \\ -1, & -1 < x < 0 \\ 0, & x = 0 \\ 1, & 0 < x < 1 \\ 0, & x \ge 1 \end{cases}$

Sketch the graph of f, find the Fourier integral representation, and deduce that $\int_0^{\infty} \frac{\sin^3 \lambda}{\lambda} d\lambda = \frac{\pi}{4}$.

Exe 1 :- Q_1 a)

Solve the following D. Eq. :

$$y'' + 5y' + 6y = 2e^{-3x} + 4e^{-2x}$$

solution :-

The auxiliay equation related to the homogeneous versionis

$$m^2 + 5m + 6 = 0 \iff (m+2)(m+3) = 0$$

$$\boxed{\Rightarrow m_1 = -2, \ m_2 = -3.}$$

$$\boxed{\Rightarrow y_c = c_1 e^{-2x} + c_2 e^{-3x}.}$$

Now $f(x) = 2e^{-3x} + 4e^{-2x}$, whence the form of y_p is

$$\boxed{y_p = Axe^{-2x} + Bxe^{-3x}.}$$

$$\Rightarrow y_p' = -2Axe^{-2x} + Ae^{-2x} - 3Bxe^{-3x} + Be^{-3x}$$

$$y_p'' = -2Ae^{-2x} + 4Axe^{-2x} - 2Ae^{-2x} - 3Be^{-3x} + 9Bxe^{-3x} - 3Be^{-3x}.$$

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Substituting in the D. Eq. we get :

$$(A\ \cancel{\quad})e^{-2x} + (-B\ \cancel{\quad})e^{-3x} = 4e^{-2x} + 2e^{-3x}$$

$$\boxed{\Rightarrow A = 4, \ B = -2.}$$

Thus $\boxed{y_p = 4xe^{-2x} - 2xe^{-3x}}$

and $y = c_1 e^{-2x} + c_2 e^{-3x} + 4xe^{-2x} - 2xe^{-3x}$

$$\boxed{\Rightarrow y = (c_1 + 4x)e^{-2x} + (c_2 - 2x)e^{-3x}.}$$

$$y(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$$

$$y'(x) = 4e^{-2x} - 2(c_1 + 4x)e^{-2x} - 2e^{-3x} - 3(c_2 - 2x)e^{-3x}$$

$$y'(0) = 4 - 2c_1 - 2 - 3c_2 = 0 \Rightarrow c_2 = 2, \text{ and } c_1 = -2$$

$$y = (-2 + 4x)e^{-2x} + (2 - 2x)e^{-3x}$$

Q1 b)

③ find the general solution of the differential equation

$$\left(\frac{1}{x} + 6xy + 4xye^{x^2}\right)dx + \left(\frac{1}{y} + 3x^2 + 2e^{x^2}\right)dy = 0$$

<u>solution</u>

$M = \frac{1}{x} + 6xy + 4xye^{x^2}$

$N = \frac{1}{y} + 3x^2 + 2e^{x^2}$

$\frac{\partial M}{\partial y} = 6x + 4xe^{x^2}$ $\Rightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\Rightarrow$ the equation is exact

$\frac{\partial N}{\partial x} = 6x + 4xe^{x^2}$

so

$f(x,y) = \int M\,dx = \int\left(\frac{1}{x} + 6xy + 4xye^{x^2}\right)dx$

$= \ln|x| + 3x^2y + 2ye^{x^2} + g(y)$

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$\frac{\partial f(x,y)}{\partial y} = 3x^2 + 2e^{x^2} + g'(y) = N$

$\Rightarrow g'(y) = \frac{1}{y} \Rightarrow g(y) = \ln|y| + C$

$\Rightarrow f(x,y) = \ln|x| + 3x^2y + 2ye^{x^2} + \ln|y| + C$

$f(x,y) = \ln x + 3x^2y + 2ye^{x^2} + \ln y + \underline{C = 0}$

Q_2 a) Ch. Eq $m^2-6m+9=0 \Rightarrow (m-3)^2=0 \Rightarrow m_1=m_2=3$

$$y_{gh} = c_1 e^{3x} + c_2 x e^{3x}$$

$y_p = x^2 A e^{3x}$, $y_p' = 2x A e^{3x} + 3x^2 A e^{3x}$

$y_p'' = 2A e^{3x} + 6x A e^{3x} + 6x A e^{3x} + 9x^2 A e^{3x}$

Hence $\cancel{9x^2 A} + \cancel{12xA} + 2A - \cancel{12xA} - \cancel{18x^2 A} + \cancel{9Ax^2} = 1$

$$\Rightarrow \quad A = \frac{1}{2} \Rightarrow y_p = \frac{1}{2} x^2 e^{3x}$$

$$y_g = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{2} x^2 e^{3x}$$

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$y(0) = c_1 = 1$

$y(1) = c_1 \cancel{e^3} + c_2 \cancel{e^3} + \frac{1}{2} \cancel{e^3} = \cancel{e^3}$

$$\Rightarrow \quad c_2 = -\frac{1}{2}$$

Thus $y_g = e^{3x} - \frac{1}{2} x e^{3x} + \frac{1}{2} x^2 e^{3x}$

$$y_g(2) = e^6 - e^6 + 2e^6 = 2e^6$$

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Q_2 b) $y'' + x y' + x^2 y = 0 \longrightarrow (*)$

The functions $\frac{a_1(x)}{a_2(x)} = x$, $\frac{a_0(x)}{a_2(x)} = x^2$ are analytic for all $x \in \mathbb{R}$

then $x_0 = 0$ is an ordinary point. The sol is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

By substitution in $(*)$, we have

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow 2a_2 + (6a_3 + a_1)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + a_{n-2}\right] x^n = 0$$

$$\Rightarrow \boxed{a_2 = 0}, \quad 6a_3 + a_1 = 0 \Rightarrow \boxed{a_3 = -\frac{a_1}{6}}$$

$$a_{n+2} = -\frac{n a_n + a_{n-2}}{(n+2)(n+1)}, \quad \forall n \geq 2$$

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$\underline{n=2}$: $a_4 = -\frac{2a_2 + a_0}{12} = \frac{-a_0}{12}$

$\underline{n=3}$: $a_5 = -\frac{1}{20}(3a_3) - \frac{1}{20} a_1 = \left(\frac{-3}{20}\right)\left(\frac{-a_1}{6}\right) - \frac{a_1}{20} = -\frac{a_1}{40}$

Thus $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$= a_0 + a_1 x - \frac{a_1}{6} x^3 - \frac{a_0}{12} x^4 - \frac{a_1}{40} x^5 + \cdots$

$= a_0 \left(1 - \frac{1}{12} x^4 + \cdots\right) + a_1 \left(x - \frac{x^3}{6} - \frac{x^5}{40} + \cdots\right)$

$= a_0 y_1 + a_1 y_2$

Exo 2 :- Q3 a)

Find only the form of the particular solution of the DEq.

$$y'' - 4y' + 4y = x e^{2x} + x^3 e^{2x} + e^{2x} \sin 2x$$

Solutions :

The auxiliary equation is $m^2 - 4m + 4 = 0 \rightarrow \boxed{m_1 = m_2 = 2}$

$m = 2$ is a root of multiplicity 2.

Now $f(x) = (x^3 + x) e^{2x} + e^{2x} \sin 2x$

Thus

$$y_p = x^2 (Ax^3 + Bx^2 + Cx + D) e^{2x} + e^{2x} (E \sin 2x + F \cos 2x).$$

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Q3 b) $m^2(m^4-16)(m-3)^2=0$

The roots are $0, 0, 2, -2, 2i, -2i, 3, 3$

$y_1 = 1,\ y_2 = x,\ y_3 = e^{2x},\ y_4 = e^{-2x},\ y_5 = \cos(2x),\ y_6 = \sin(2x)$

$y_7 = e^{3x},\ y_8 = x e^{3x}$

These solutions are linearly independent on $\mathbb{R}$.

$$y_{gh} = C_1 + C_2 x + C_3 e^{2x} + C_4 e^{-2x} + C_5 \cos(2x) + C_6 \sin(2x) + C_7 e^{3x} + C_8 x e^{3x}.$$

(Q3 c)

Exo 3 :

solve the following Cauchy-Euler D.Eq.

$$x^2y'' - xy' - 3y = x^2 \ln x$$

solution

put $\boxed{x = e^t}$, then $t = \ln x$, $xy' = \frac{dy}{dt}$, $x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

Substituting in the D.Eq, we get

$$\boxed{\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = te^{2t} \quad (*)}$$

The auxiliary equation of $(*)$ is $m^2 - 2m - 3 = 0$

So $m_1 = -1$, $m_2 = 3$, whence

$$\boxed{y_c(t) = c_1 e^{-t} + c_2 e^{3t}.}$$

$$y_p = (At + B)e^{2t} \Rightarrow \begin{cases} y_p' = Ae^{2t} + 2(At+B)e^{2t} \\ y_p'' = 2Ae^{2t} + 2Ae^{2t} + 4(At+B)e^{2t} \end{cases}$$

Substituting in $(*)$, we get

$$(2A - 3B - 3At)e^{2t} = te^{2t} \Rightarrow \begin{cases} 2A - 3B = 0 \\ -3A = 1 \end{cases}$$

$$\Rightarrow A = -\frac{1}{3}, \quad B = -\frac{2}{9}$$

$$\Rightarrow \boxed{y_p(t) = \left(-\frac{1}{3}t - \frac{2}{9}\right)e^{2t}}$$

$$\boxed{y(t) = c_1 e^{-t} + c_2 e^{3t} - \left(\frac{1}{3}t + \frac{2}{9}\right)e^{2t}}$$

But $e^t = x$, therefore we obtain :

$$\boxed{y = c_1 x^{-1} + c_2 x^3 - \left(\frac{1}{3}\ln x + \frac{2}{9}\right)x^2}$$

Remark : The Sublent can solve it directly using Variation of parameters.

Q1 Find the Fourier series for the periodic function

$$f(x) = \pi - |x|, \quad -\pi \le x \le \pi$$

of period 2π and show that

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Solution. The given function is an even function and therefore its Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

where, we find

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (\pi - x)dx = \pi$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx dx = \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right].$$

Hence, the Fourier series is

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx.$$

~~On substituting $x = 0$ in above expression, we get the desired result.~~

Q4 a)

Q2. Find the Fourier sine series for the periodic function

$$f(x) = 1 + x, \quad 0 \le x \le 1$$

of period 2 and show that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Solution. The Fourier sine series of the function is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x,$$

where

$$b_n = 2\int_0^1 (1+x)\sin n\pi x dx = \frac{2}{\pi}\left[\frac{1-2(-1)^n}{n}\right].$$

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Thus, the Fourier sine series is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]\sin n\pi x.$$

Choosing $x = \frac{1}{2}$ in above series, we get the desired result.

Q3. Find the Fourier integral for the function

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 0 \end{cases},$$

and deduce that

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

Solution. Note that the function is an even function and therefore its Fourier integral is

$$f(x) = \int_0^\infty A(\alpha)\cos\alpha x d\alpha,$$

where

$$A(\alpha) = \frac{2}{\pi}\int_0^1 \cos\alpha x dx = \frac{2}{\pi}\frac{\sin\alpha}{\alpha}.$$

Thus, the Fourier integral is

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha}{\alpha}\cos\alpha x d\alpha$$

and on taking $x = 0$ in above integral, we get the desired result.

Q4 b)

Q4. Find the Fourier integral of the function

$$f(x) = \begin{cases} 0, & x < -1 \\ -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \\ 0, & x > 0 \end{cases}$$

and deduce that

$$\int_0^\infty \frac{\sin^3\lambda}{\lambda}d\lambda = \frac{\pi}{4}.$$

Solution. We notice that the given function is an odd function and its Fourier integral is

$$f(x) = \frac{1}{\pi}\int_0^\infty B(\alpha)\sin\alpha x d\alpha,$$

(4 b)

$$2\int_0^1 \sin \alpha x \, dx = \frac{2}{\alpha}\left(1 - \cos \alpha\right)$$

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where

$$B(\alpha) = \frac{2}{\pi}\int_0^1 \sin \alpha x dx = \frac{2}{\pi}\frac{2\,(1-\cos\alpha)}{\alpha}.$$

Thus, the Fourier integral is

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos\alpha}{\alpha}\right)\sin \alpha x d\alpha.$$

Choosing $x = \frac{1}{2}$ in above equation, we have

$$1 = \frac{4}{\pi}\int_0^\infty \left(\frac{\sin^2\frac{\alpha}{2}}{\alpha}\right)\sin\frac{\alpha}{2}d\alpha,$$

which on substitution $2\lambda = \alpha$, we get the desired result.

~~Q5. Solve the system of equations~~

$$2\frac{dx}{dt} - \frac{dy}{dt} - x + 2y = 0$$

$$2\frac{dy}{dt} + x - y = 0$$

Solution: Given system is

$$(2D-1)x + (-D+2)y = 0, \quad x + (2D-1)y = 0.$$

On eliminating x, we get

$$\left[(2D-1)^2 - (-D+2)\right]y = 0,$$

that is,

$$(4D^2 - 3D - 1)\,y = 0.$$

Solving this constant coeficient equation, we get,

$$y(t) = c_1 e^t + c_2 e^{\frac{-t}{4}}$$

and the second equation in the system gives

$$x(t) = -c_1 e^t + \frac{3}{4}c_2 e^{\frac{-t}{4}}.$$

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