

Time: 3 hours

Signature:

Question 1 [Marks 10]: Which of the given choices are correct?

- (i) If square of a matrix A is zero matrix, then $I - A$ is equal to:
a) 0 b) $(A-I)^{-1}$ c) $(A+I)^{-1}$ d) $A + I$
- (ii) If A is a square matrix of order 3 with $\det(A) = 2$, then $\det(\det(\frac{1}{\det(A)} A^3) A^{-1})$ is equal to:
a) $1/4$ b) $1/2$ c) $1/3$ d) $1/16$
- (iii) If the general solution of $AX = O$ is $(-2r, 4r, r)$, $r \in \mathbb{R}$, and $(1, 0, -2)$ is a solution of $AX = B$, then the general solution of $AX = B$ is:
a) $(1 - 2r, 4r, r - 2)$ b) $(-2r, 0, -2r)$ c) $(-2r, 4r, r)$ d) $(-2r - 1, 4r, r - 2)$
- (iv) A subset S of $\mathbb{R}^3$ is a basis of the vector space $\mathbb{R}^3$ if S is equal to:
a) $\{(1, 0, 0), (0, 2, 1), (0, 6, 0)\}$ b) $\{(1, 1, 0), (2, 1, 0), (3, 2, 0)\}$ c) $\{(1, 1, 0), (0, 0, 0), (3, 2, 1)\}$ d) $\{(1, 1, 0), (0, 0, 1), (2, 2, 1)\}$
- (v) If $B = \{u_1 = (2, 1), u_2 = (4, 3)\}$ and $C = \{v_1 = (0, 1), v_2 = (6, 0)\}$ are ordered bases of $\mathbb{R}^2$, then the transition matrix $P_{C \rightarrow B}$ from C to B is equal to:
a) $\begin{bmatrix} 1 & -1/2 \\ -2 & 3/2 \end{bmatrix}$ b) $\begin{bmatrix} -2 & 9 \\ 1 & -3 \end{bmatrix}$ c) $\begin{bmatrix} -2/3 & 3 \\ 1/3 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$
- (vi) If B is a square matrix of order 3 with $\det(B) = 2$, then $\text{nullity}(B)$ is equal to:
a) 2 b) 1 c) 3 d) 0
- (vii) If $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$ and $u, v \in \mathbb{R}^n$ such that $\|u\|^2 = 5$, $\|v\|^2 = 1$, $\langle u, v \rangle = -2$, then $\langle u + 2v, 5u - v \rangle$ is equal to:
a) $\sqrt{5}$ b) 5 c) 9 d) 41
- (viii) If $S = \{A, I_2\} \subseteq M_{2 \times 2}(\mathbb{R})$, where A is a non-symmetric matrix, then S must be:
a) linearly dependent b) a spanning set for $M_{2 \times 2}(\mathbb{R})$ c) linearly independent d) orthogonal
- (ix) Let T be the transformation from the Euclidean space $\mathbb{R}^2$ to $\mathbb{R}$ given by $T(u) = \|u\|$ for all $u \in \mathbb{R}^2$, where $\|u\|$ is the Euclidean norm of u . Then, for $v, w \in \mathbb{R}^2$ and $k \in \mathbb{R}$, T satisfies:
a) $T(u + v) = T(u) + T(v)$ b) $T(u + v) \leq T(u) + T(v)$ c) $T(0) > 0$ d) $T(ku) = kT(u)$
- (x) Zero is an eigenvalue of the matrix $\begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix}$ with geometric multiplicity equal to:
a) 1 b) 2 c) 3 d) 4

Question 2 [Marks 2 + 2 + 3]:

- (a) Find the square matrix A of order **3** such that $A^{-1}(A - I) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and evaluate $\det(A)$.
- (b) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \\ -2 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & -2 \\ 1 & -1 & -2 \end{bmatrix}$. Find a matrix X that satisfies $XA = B$.
- (c) Solve the following system of linear equations:

$$\begin{aligned} x + y + z &= 1 \\ 2x &+ 2z = 3 \\ 3x + 5y + 4z &= 2. \end{aligned}$$

Question 3 [Marks 3 + 3 + 2]:

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Then:

- (a) Find a basis and the dimension for each of the vector spaces $\text{row}(A)$, $\text{col}(A)$, and $N(A)$.
- (b) Decide with justification whether the following statements are true or false:
- (i) $\text{row}(A) = \text{row}(B)$ (ii) $\text{col}(A) = \text{col}(B)$ (iii) $N(A) = N(B)$.
- (c) Find all square matrices Z of order **3** such that $AZ = O$.

Question 4 [Marks 3 + (1 + 3)]:

- (a) Construct an orthonormal basis C of the Euclidean space $\mathbb{R}^3$ by applying the Gram-Schmidt algorithm on the given basis $B = \{v_1 = (1,1,0), v_2 = (1,0,1), v_3 = (0,1,1)\}$, and then find the coordinate vector of $v = (1,2,0) \in \mathbb{R}^3$ relative to the orthonormal basis C .
- (b) Let $\mathcal{P}_2$ denote the vector space of real polynomials with degree ≤ 2 . Consider the linear transformation $T: \mathbb{R}^3 \rightarrow \mathcal{P}_2$ defined by: $T(1,0,0) = x^2 + 1$, $T(0,1,0) = 3x^2 + 2$, $T(0,0,1) = -x^2$. Then:
- (i) Compute $T(a, b, c)$, for all $(a, b, c) \in \mathbb{R}^3$.
- (ii) Find a basis for each of the vector spaces $\text{Im}(T)$ and $\ker(T)$.

Question 5 [Marks 3 + 2 + 3]: Let $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix}$. Then:

- (a) Find the eigenvalues of A .
- (b) Find algebraic and geometric multiplicities of all the eigenvalues of A .
- (c) Is the matrix A diagonalizable? If yes, find a matrix P that diagonalizes A .