King Saud University
College of Sciences
Department of Mathematics
Semester 461 / Final Exam / MATH-244 (Linear Algebra)

Max. Marks: 40 **Time: 3 hours**

Name: **ID:** **Section:** **Signature:**

Note: ***Attempt all the five questions. Scientific calculators are not allowed.***

Question 1 [Marks 1×10]**:** Choose the correct answer:

(i) If the rows of a 3×4 matrix are linearly dependent, then the maximum dimension for $col(A)$ is:

(a) 1 (b) 2 (c) 3 (d) 4

(ii) If A is a nonzero 4×7 matrix, then the possible values for $nullity(A)$ are:

(a) 2,3,4,5,6 (b) 3,4,5,6,7 (c) 3,4,5,6 (d) 1,2,3,4

(iii) If u and v are nonzero vectors in an inner product space with $d(u,v) = d(u,-v)$, then u is orthogonal to:

(a) u (b) $u+v$ (c) $u-v$ (d) v.

(iv) If $\{u,v\}$ is linearly independent and $\{u,v,w\}$ is linearly dependent, then:

(a) $\{u,w\}$ is linearly independent. (b) $\{v,w\}$ is linearly independent. (c) $w\in span\{u,v\}$ (d) $u\in span\{v,w\}$.

(v) If $\begin{bmatrix}1 & 2\\ 3 & -3\end{bmatrix}$ is the transition matrix from the basis $\{u,(1,1)\}$ to the basis $\{(5,4),v\}$ for $\mathbb{R}^2$, then u is equal to:

(a) $(1,3)$ (b) $(4,-1)$. (c) $(14,11)$ (d) $(3,0)$.

(vi) If $S=\{v_1=(1,1), v_2=(1,0)\}$ is a basis for $\mathbb{R}^2$ and the transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ is such that $T(v_1)=(1,-2)$ and $T(v_2)=(-4,1)$, then $T(5,-3)$ equals:

(a) (-35, 14) (b) (2, -10) (c) (2, 5) (d) (2, 10).

(vii) The set $\{(-3, 4, 0), (4, x, 0), (0, 0, x)\}$ of vectors in the Euclidean space $\mathbb{R}^3$ is orthogonal *iff* :

(a) $x = 1$ (b) $x = 3$ (c) $x = -3$ (d) $x = 5$

(viii) If $T:\mathbb{R}^2\to\mathbb{R}^2$ is a linear transformation with $T(1,0)=(1,2)$ and $T(1,1)=(5,-3)$, then its standard matrix is:

(a) $\begin{bmatrix}1 & 4\\ 2 & -5\end{bmatrix}$ (b) $\begin{bmatrix}1 & 5\\ 2 & -3\end{bmatrix}$ (c) $\begin{bmatrix}1 & 2\\ 5 & -3\end{bmatrix}$ (d) $\begin{bmatrix}1 & 6\\ 2 & -1\end{bmatrix}$

(ix) The eigenvalues of a square matrix A are the same as the eigenvalues of:

(a) A^2. (b) A^T. (c) $RREF(A)$. (d) $adj(A)$

(x) If $\boldsymbol{A}$ is diagonalizable matrix, then $det(\boldsymbol{A})$ equals:

(a) *The sum of the eigen values of* $\boldsymbol{A}$ (b) *The product of the eigen values of* $\boldsymbol{A}$ (c) *Zero* (d) *Number of columns in* $\boldsymbol{A}$.

Question 2 [Marks 2 + 2 + 2]**:**

(a) Let a matrix $\boldsymbol{A}$ satisfy $\boldsymbol{A}^2 + \boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Then show that A is invertible.

(b) Consider $\boldsymbol{B}, \boldsymbol{C} \in \mathbf{M}_3(\mathbb{R})$ with $|\mathbf{B}| = 2|\mathbf{C}| = 1$. Then evaluate $|3\, \boldsymbol{C}\, \mathbf{B}\, adj(B^{-3})|$.

(c) Find the values of *a*, *b* such that the following system of linear equations

$$x - 2y + 3z = 4$$
$$3x - 4y + 5z = b$$
$$2x - 3y + az = 5$$

has: (i) no solution (ii) unique solution.

Question 3 [Marks 3 + 3 + 2]**:**

(a) Find a subset **B** of **G** = $\{(1,1,-4,-3), (2,0,2,-2), (1,2,-9,-5)\}$ that forms a basis for *span*(**G**). Then express each vector in $\boldsymbol{G} - \boldsymbol{B}$ as a linear combination of vectors in **B**.

(b) Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$. Then find a basis for the column space *col*($\boldsymbol{A}$) and dimension of the null space *N*($\boldsymbol{A}$).

(c) Let B and $\boldsymbol{B'}$ be two ordered bases for $\mathbb{R}^2$ with a transition matrix $\boldsymbol{P}_{\mathrm{B}\to\mathrm{B}'} = \begin{bmatrix} 5 & 3 \\ -1 & -1 \end{bmatrix}$ from $\boldsymbol{B}$ to $\boldsymbol{B'}$. If $[\boldsymbol{v}]_{\mathbf{B}'} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$ is the coordinate vector of a vector $\boldsymbol{v} \in \mathbb{R}^2$ relative to the basis $\boldsymbol{B}'$. Then find $[\boldsymbol{v}]_{\boldsymbol{B}}$.

Question 4 [Marks 2 + 3 + 5]**:**

(a) Let $\{v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_3 = (0,0,1,0), v_4\}$ be the orthonormal basis obtained by applying the Gram-Schmidt algorithm on the basis $\{u_1 = (3,0,0,0), u_2 = (3,3,0,0), u_3 = (3,3,3,0), u_4 = (3,3,3,3)\}$ of Euclidean inner product space $\mathbb{R}^4$. Then find the vector v_4.

(b) Let v_0 be any fixed vector in an inner product space V of dimension n and $T: V \to \mathbb{R}$ be the linear transformation defined by $T(\mathrm{v}) = < v, v_0 >$ for all $v \in V$. If $v_0 \in Ker(T)$, then show that $nullity(T) = n$.

(c) Let $\boldsymbol{B} = \{u_1 = (1,0,0), u_2 = (0,1,0), u_3 = (0,0,1)\}$ and $\boldsymbol{C} = \{v_1 = (1,1,1)), v_2 = (1,1,0), v_3 = (1,0,0)\}$ be two ordered bases for $\mathbb{R}^3$. Let the linear transformation $\boldsymbol{T}: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by:

$$\boldsymbol{T}(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 - x_1, x_1 - x_3)\}.$$

Find the matrix $[\boldsymbol{T}]_{\boldsymbol{B}}$ of the transformation $\boldsymbol{T}$ relative to the basis $\boldsymbol{B}$ and then use it to find the matrix $[\boldsymbol{T}]_{\boldsymbol{C}}$.

Question 5 [Marks 2 + 1 + 3]**:** Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$. Then:

(a) Find the eigenvalues of $\boldsymbol{A}$.

(b) Is the matrix $\boldsymbol{A}$ diagonalizable? Justify your answer.

(c) Find a diagonal matrix $\boldsymbol{D}$ and an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{D}$.

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