

Exercises

April 22, 2025

1 Real Numbers

1. Determine $\sup A$, $\inf A$, $\max A$, $\min A$ where they exist;

(a) $A = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

(b) $A = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\right\}$

(c) $A = \{x \in \mathbb{R} : x^2 - 4 < 0\}$

2. If b is an upper bound of A , prove that $b = \sup A$ if and only if for every $\varepsilon > 0$ there is an element $a \in A$ such that $a > b - \varepsilon$
3. If the sets A and B are bounded above and $A \subseteq B$, prove that

$$\sup A \leq \sup B$$

4. Let A and B be bounded subsets of $\mathbb{R}$, and define

$$A + B = \{a + b : a \in A, b \in B\}$$

Prove that $\sup(A + B) = \sup A + \sup B$

2 Sequences

2.1 convergent sequences

1. Use the definition to show that

(a) $\lim_{n \rightarrow \infty} \frac{2n-1}{3n+2} = \frac{2}{3}$

(b) $\lim_{n \rightarrow \infty} \frac{n^3+1}{2n^3+n} = \frac{1}{2}$.

2. Prove that $\lim x_n = 0$ if and only if $\lim |x_n| = 0$
3. If the two sequences (x_n) and (y_n) converge to c , and we define the shuffled sequence (z_n) by

$$(z_1, z_2, z_3, z_4, \dots) = (x_1, y_1, x_2, y_2, \dots)$$

show that the sequence (z_n) also converges to c

4. Give an example of two sequences (x_n) and (y_n) such that $(x_n + y_n)$ is convergent and (x_n) is divergent.
5. If the sequences (x_n) and $(x_n + y_n)$ are both convergent, prove that (y_n) is also convergent and determine its limit.
Can you state a corresponding result for the sequence $(x_n \cdot y_n)$?
6. Give an example of a divergent sequence (x_n) such that $(|x_n|)$ is convergent.
When does the convergence of $(|x_n|)$ imply the convergence of (x_n) , and what is the relation between $\lim |x_n|$ and $\lim x_n$ when both exist?
7. If $\lim \frac{x_n - 1}{x_n + 1} = 0$, prove that $\lim x_n = 1$

2.2 Monotonic sequences

8. Prove that $x_n = \frac{n^n}{n!}$ is monotonic
9. Prove that each of the following sequences is monotonic and bounded, then find its limit
 - (a) $x_1 = 1, x_{n+1} = \sqrt{3 + x_n}$, for all $n \in \mathbb{N}$
 - (b) $x_1 = 1, x_{n+1} = \frac{4x_n + 2}{x_n + 3}$, for all $n \in \mathbb{N}$
10. Given $x_n = \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n}$, prove that (x_n) is decreasing and bounded then conclude it is convergent.

2.3 Cauchy criterion

11. Show by definition that (x_n) is a Cauchy sequence where

$$x_n = \frac{5n}{n+3}$$

12. If (x_n) satisfies

$$|x_{n+1} - x_n| < \frac{1}{2^n}$$

prove that (x_n) is a Cauchy Sequence.

13. If

$$x_n = \sum_{k=1}^n \frac{1}{k^2}$$

prove that (x_n) is a Cauchy sequence.

2.4 Subsequences

14. Prove that the sequence (x_n) has a convergent subsequence

$$x_n = \frac{(n^2 + 20n + 30) \sin(n^3)}{n^2 + n + 1}$$

15. Give an example of

- (a) a sequence with no convergent subsequence
- (b) an unbounded sequence which has a convergent subsequence

16. If every subsequence of (x_n) has a subsequence which converges to 0, prove that $\lim x_n = 0$

3 Series

1. Prove that changing a finite number of terms of a series does not affect its convergence.
2. If $x_n \geq 0$ for all $n \in \mathbb{N}$, prove that the series $\sum x_n$ converges if, and only if, the sequence $S_n = \sum_{k=1}^n x_k$ is bounded above
3. Prove that if $a > 0$, then

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)(n+1+a)} = \frac{1}{a}$$

4. Test the following series for convergence

(a)

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+1)}$$

(b)

$$\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$$

(c)

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

(d)

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

5. Prove that the convergence of $\sum a_n^2$ and $\sum b_n^2$ implies the convergence of $\sum a_n b_n$
 6. Prove that the convergence of $\sum a_n^2$ implies the convergence of $\sum \frac{a_n}{n}$
 7. Let the sequence (x_n) be positive and decreasing. If $\sum x_n$ converges, prove that $\lim nx_n = 0$
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4 Limits of Functions

1. Prove by definition that

(a) $\lim_{x \rightarrow 2} x^3 = 8$

(b) $\lim_{x \rightarrow 4} \sqrt{x} = 2$

2. Prove that the following limits does not exist

(a) $\lim_{x \rightarrow 0^+} \frac{1}{\sqrt{x}}$

(b) $\lim_{x \rightarrow 0} \cos \frac{1}{2x}$

3. Let $f : D \rightarrow \mathbb{R}$ and $c \in \hat{D}$. If $\lim_{x \rightarrow c} f(x)^2 = 0$, prove that $\lim_{x \rightarrow c} f(x) = 0$. Give an example of a function f for which $\lim_{x \rightarrow c} f(x)^2$ exists but $\lim_{x \rightarrow c} f(x)$ does not exist.

4. Give an example of two functions such that

(a) $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ do not exist, but $\lim_{x \rightarrow c} (f(x) + g(x))$ exists.

(b) $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ do not exist, but $\lim_{x \rightarrow c} f(x)g(x)$ exists.

5. If $\lim_{x \rightarrow c} f(x)$ exists and $\lim_{x \rightarrow c} (f(x) + g(x))$ does not exist, what we can say about $\lim_{x \rightarrow c} g(x)$
6. Determine the values of the following limits where they exist:

- (a) $\lim_{x \rightarrow 0} \frac{x^2}{|x|}$
- (b) $\lim_{x \rightarrow 0} \frac{x^2(3 + \sin x)}{(x + \sin x)^2}$
7. If $\lim_{x \rightarrow c} f(x) = 0$ and g is a bounded function in some neighborhood of c , prove that $\lim_{x \rightarrow c} f(x)g(x) = 0$.
8. If $f(x) \geq 0$ for all $x \in D_f$, and $\lim_{x \rightarrow c} f(x) = L$, prove that $\lim_{x \rightarrow c} \sqrt{f(x)} = \sqrt{L}$.
9. If $\lim_{x \rightarrow c} f(x) = L$, prove that $\lim_{x \rightarrow c} |f(x)| = |L|$. When is the converse also true?
10. Prove by definition that $\lim_{x \rightarrow 3} \frac{1}{(x-3)^2} = \infty$
11. Determine the values of the following limits where they exist:
- (a) $\lim_{x \rightarrow \infty} \frac{2x^3 + 2x + 1}{3x^3 + x^2}$
- (b) $\lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 + 1}}$
12. Let $f : (0, \infty) \rightarrow \mathbb{R}$. Prove that $\lim_{x \rightarrow 0^+} f\left(\frac{1}{x}\right) = L$ if and only if $\lim_{x \rightarrow \infty} f(x) = L$.

5 Continuity

1. Let $f : (-1, 1) \rightarrow \mathbb{R}$ satisfy

$$|f(x)| \leq |x| \quad \forall x \in (-1, 1)$$

Show that f is continuous at $x = 0$.

2. If the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $f(x) = 0$ for all $x \in \mathbb{Q}$, prove that $f(x) = 0$ for all $x \in \mathbb{R}$.
3. Give an example of two functions, one of which is not continuous at c , whose product is continuous at c .
4. Give an example of two functions, one of which is not continuous at c , whose composition is continuous at c .
5. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous and $f(a) < g(a)$, $f(b) > g(b)$. Show that there is $c \in (a, b)$ such that $f(c) = g(c)$.

6. Prove that the equation $\cos x = x$ has a solution in $(0, \pi/2)$.
 7. Determine a real interval of length 1 where the equation $x^4 - 2x^3 - 5 = 0$ has a solution.
 8. prove that if $f : [a, b] \rightarrow \mathbb{R}$ is continuous and has a maximum at $c \in (a, b)$ then f is not injective.
 9. Determine which functions are uniformly continuous:
 - (a) $f(x) = \sqrt[3]{x}, \quad x \in \mathbb{R}$
 - (b) $f(x) = \frac{x^2}{x+1}, \quad x \in (0, \infty)$
 - (c) $f(x) = x \sin x, \quad x \in \mathbb{R}$
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6 Differentiation

6.1 The Derivative

1. Use the definition to find the derivative of $f(x) = \frac{1}{x}, x \neq 0$
2. Find the set of points where the function f is not differentiable
 - (a) $f(x) = |x^2 - 1|$
 - (b) $f(x) = x|x|$
3. If $g(0) = g'(0) = 0$, find $f'(0)$ where

$$f(x) = \begin{cases} g(x) \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

4. Let

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c \end{cases}$$

Prove that f is differentiable at $x = 0$, and evaluate $f'(0)$.

5. If the function f satisfies $|f(x)| \leq |x|^r$, where $r > 1$, prove that f is differentiable at $x = 0$.
6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$. The function f is even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$, and odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. If f is differentiable, prove that f' is odd when f is even, and even when f is odd.

6.2 Mean Value Theorem

7. Use the definition to show that $f(x) = \sqrt{x^2 + 1}$ is differentiable on $\mathbb{R}$, then prove that there is $c \in (0, 1)$ such that

$$\sqrt{2} - 1 = \frac{c}{\sqrt{c^2 + 1}}$$

8. If $f : \mathbb{R} \rightarrow \mathbb{R}$, and there is a real constant $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|^2 \quad \forall x, y \in \mathbb{R}$$

show that f is constant.

9. Prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
10. Prove that

$$\sqrt{1+x} < 1 + \frac{x}{2} \quad \forall x > 0$$

6.3 L'Hopital's Rule

11. Evaluate the following limits

(a) $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2}$

(b) $\lim_{x \rightarrow 0^+} \left(1 + \frac{2}{x}\right)^x$

12. Let $g \in C^2(\mathbb{R})$ such that $g(0) = g'(0) = 0$ and $g''(0) = 6$. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and defined by

$$f(x) = \frac{g(x)}{x}, \quad x \neq 0$$

Find $f(0)$, and discuss the differentiability of f at $x = 0$.

13. If $f(x) = x^2 \sin(\frac{1}{x})$, $g(x) = \sin x$, show that $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)}$ exists, while $\lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)}$ does not exist.

6.4 Taylor's Theorem

14. Prove that for $x > 0$,

$$1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{x+1} \leq 1 + \frac{x}{2}$$

15. Decide whether $f(0)$ is an extremum value of $f(x) = \sin x - x + \frac{x^3}{6}$.

7 Reimann Integral

1. Determine whether f is integrable over $[0, 1]$ and evaluate $\int_a^b f(x)dx$ whenever it exists.

(a)

$$f(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ 0 & x \in [\frac{1}{2}, 1] \end{cases}$$

(b)

$$f(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}) \\ x - 2 & x \in [\frac{1}{2}, 1] \end{cases}$$

(c)

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c \end{cases}$$

- (d) $f(x) = x^2$. [Hint: Use the formula $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$]

2. Let f be continuous and non-negative on $[a, b]$. If $\int_a^b f(x)dx = 0$, prove that $f(x) = 0$ for all $x \in [a, b]$.

3. If $f \in \mathcal{R}(0, 1)$, prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x)dx$$

4. Use Reimann sum to evaluate $\int_1^5 (3x + 7)dx$

5. If f is bounded on $[a, b]$ and $k > 0$, prove that

$$L(kf) = kL(f), \quad U(kf) = kU(f)$$

What if $k \leq 0$? Deduce that if $f \in \mathcal{R}(a, b)$, then for any real number k ,

$$kf \in \mathcal{R}(a, b) \text{ and } \int_a^b kf = k \int_a^b f$$

6. Let f and g be bounded on $[a, b]$. If $f \leq g$, prove that

$$L(f) \leq L(g), \quad U(f) \leq U(g)$$

7. Give an example of a function $f \notin \mathcal{R}(a, b)$ but $|f| \in \mathcal{R}(a, b)$.

8. If f is continuous on $[a, b]$ and H is given by $H(x) = \int_x^b f(t)dt$, calculate $H'(x)$.

9. Suppose f is continuous on $[a, b]$ and $g, h : [c, d] \rightarrow [a, b]$ are differentiable. If $F : [c, d] \rightarrow \mathbb{R}$ is defined by

$$F(x) = \int_{h(x)}^{g(x)} f(t) dt,$$

prove that

$$F'(x) = f(g(x))g'(x) - f(h(x))h'(x)$$

10. If $F(x) = \int_{2x}^{x^2} \sin \sqrt{1+t^2} dt$, find $F'(x)$
11. Evaluate the following improper integrals:

(a) $\int_0^1 \log x dx$

(b) $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

8 Sequences and Series of Functions

1. Determine the pointwise limit of (f_n) , then decide whether the convergence is uniform or not.

(a) $f_n(x) = \frac{\sin(nx)}{nx}, \quad x \in (0, 1)$

(b) $f_n(x) = \frac{1}{nx+1}, \quad x \in (0, 1)$

(c) $f_n(x) = \frac{x}{nx+1}, \quad x \in [0, 1]$

2. Find the limit of the sequence $f_n(x) = \frac{x^n}{x^n+1}$ on $[0, 2]$, and determine whether the convergence is uniform.
3. The functions f_n on $[-1, 1]$ are defined by

$$f_n(x) = \frac{x}{n^2x^2+1}$$

Show that (f_n) converges uniformly and that its limit f is differentiable, but that the equality $f'(x) = \lim f'_n(x)$ does not hold for all $x \in [-1, 1]$.

4. Determine where the series $\sum f_n$ converges pointwise and where it converges uniformly

(a) $f_n(x) = \frac{1}{x^2+n^2}$

(b) $f_n(x) = \frac{1}{x^n + 1}, \quad x \neq -1$

5. If $\sum a_n$ is absolutely convergent, prove that $\sum a_n \cos nx$ and $\sum a_n \sin nx$ uniformly convergent on $\mathbb{R}$.
6. Determine the radius of convergence of the series $\sum \frac{x^n}{n^n}$.
7. For what values of c is the series $\sum \frac{x^n}{n^{\frac{1}{n}}}$ uniformly convergent on $[-c, c]$?