Exercises

March 9, 2025

1 Real Numbers

1. Determine $\sup A, \inf A, \max A, \min A$ where they exist;
   (a) $A = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$
   (b) $A = \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$
   (c) $A = \{x \in \mathbb{R} : x^2 - 4 < 0\}$
2. If b is an upper bound of A, prove that $b = \sup A$ if and only if for every $\varepsilon > 0$ there is an element $a \in A$ such that $a > b - \varepsilon$
3. If the sets A and B are bounded above and $A \subseteq B$, prove that
$$\sup A \leq \sup B$$
4. Let A and B be bounded subsets of $\mathbb{R}$, and define
$$A + B = \{a + b : a \in A, b \in B\}$$
   Prove that $\sup(A + B) = \sup A + \sup B$

2 Sequences

2.1 convergent sequences

1. Use the definition to show that
   (a) $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$
   (b) $\lim \frac{n^3+1}{2n^3+n} = \frac{1}{2}$.

2. Prove that $\lim x_n = 0$ if and only if $\lim |x_n| = 0$

3. If the two sequences (x_n) and (y_n) converge to c, and we define the shuffled sequence (z_n) by

$$(z_1, z_2, z_3, z_4, \dots) = (x_1, y_1, x_2, y_2, \dots)$$

show that the sequence (z_n) also converges to c

4. Give an example of two sequences (x_n) and (y_n) such that $(x_n + y_n)$ is convergent and (x_n) is divergent.

5. If the sequences (x_n) and $(x_n + y_n)$ are both convergent, prove that (y_n) is also convergent and determine its limit.
Can you state a corresponding result for the sequence $(x_n \cdot y_n)$?

6. Give an example of a divergent sequence (x_n) such that $(|x_n|)$ is convergent.
When does the convergence of $(|x_n|)$ imply the convergence of (x_n), and what is the relation between $\lim |x_n|$ and $\lim x_n$ when both exist?

7. If $\lim \dfrac{x_n - 1}{x_n + 1} = 0$, prove that $\lim x_n = 1$

2.2 Monotonic sequences

8. Prove that $x_n = \frac{n^n}{n!}$ is monotonic

9. Prove that each of the following sequences is monotonic and bounded, then find its limit

   (a) $x_1 = 1,\ x_{n+1} = \sqrt{3 + x_n}$, for all $n \in \mathbb{N}$

   (b) $x_1 = 1,\ x_{n+1} = \dfrac{4x_n + 2}{x_n + 3}$, for all $n \in \mathbb{N}$

10. Given $x_n = \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n}$, prove that (x_n) is decreasing and bounded then conclude it is convergent.

2.3 Cauchy criterion

11. Show by definition that (x_n) is a Cauchy sequence where

$$x_n = \frac{5n}{n+3}$$

12. If (x_n) satisfies

$$|x_{n+1} - x_n| < \frac{1}{2^n}$$

prove that (x_n) is a Cauchy Sequence.

13. If

$$x_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

prove that (x_n) is a Cauchy sequence.

2.4 Subsequences

14. Prove that the sequence (x_n) has a convergent subsequence

$$x_n = \frac{(n^2 + 20n + 30)\sin(n^3)}{n^2 + n + 1}$$

15. Give an example of

    (a) a sequence with no convergent subsequence

    (b) an unbounded sequence which has a convergent subsequence

16. If every subsequence of (x_n) has a subsequence which converges to 0, prove that $\lim x_n = 0$

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3 Series

1. Prove that changing a finite number of terms of a series does not affect its convergence.

2. If $x_n \geq 0$ for all $n \in N$, prove that the series $\sum x_n$ converges if, and only if, the sequence $S_n = \sum_{k=1}^{n} x_k$ is bounded above

3. Prove that if $a > 0$, then

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)(n+1+a)} = \frac{1}{a}$$

4. Test the following series for convergence

    (a)

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+1)}$$

    (b)

$$\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$$

(c)

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

(d)

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

5. Prove that the convergence of $\sum a_n^2$ and $\sum b_n^2$ implies the convergence of $\sum a_n b_n$
6. Prove that the convergence of $\sum a_n^2$ implies the convergence of $\sum \frac{a_n}{n}$
7. Let the sequence (x_n) be positive and decreasing. If $\sum x_n$ converges, prove that $\lim n x_n = 0$

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4 Limits of Functions

1. Prove by definition that
   (a) $\lim_{x \to 2} x^3 = 8$
   (b) $\lim_{x \to 4} \sqrt{x} = 2$
2. Prove that the following limits does not exist
   (a) $\lim_{x \to 0^+} \frac{1}{\sqrt{x}}$
   (b) $\lim_{x \to 0} \cos \frac{1}{2x}$
3. Let $f : D \to \mathbb{R}$ and $c \in \hat{D}$. If $\lim_{x \to c} f(x)^2 = 0$, prove that $\lim_{x \to c} f(x) = 0$. Give an example of a function f for which $\lim_{x \to c} f(x)^2$ exists but $\lim_{x \to c} f(x)$ does not exist.
4. Give an example of two functions such that
   (a) $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ do not exist, but $\lim_{x \to c} (f(x) + g(x))$ exists.
   (b) $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ do not exist, but $\lim_{x \to c} f(x)g(x)$ exists.
5. If $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} (f(x) + g(x))$ does not exist, what we can say about $\lim_{x \to c} g(x)$
6. Determine the values of the following limits where they exist:

(a) $\lim_{x \to 0} \frac{x^2}{|x|}$

(b) $\lim_{x \to 0} \frac{x^2(3 + \sin x)}{(x + \sin x)^2}$

7. If $\lim_{x \to c} f(x) = 0$ and g is a bounded function in some neighborhood of c, prove that $\lim_{x \to c} f(x)g(x) = 0$.

8. If $f(x) \geq 0$ for all $x \in D_f$, and $\lim_{x \to c} f(x) = L$, prove that $\lim_{x \to c} \sqrt{f(x)} = \sqrt{L}$.

9. If $\lim_{x \to c} f(x) = L$, prove that $\lim_{x \to c} |f(x)| = |L|$. When is the converse also true?

10. Prove by definition that $\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$

11. Determine the values of the following limits where they exist:

(a) $\lim_{x \to \infty} \frac{2x^3 + 2x + 1}{3x^3 + x^2}$

(b) $\lim_{x \to -\infty} \frac{x}{\sqrt{x^2 + 1}}$

12. Let $f : (0, \infty) \to \mathbb{R}$. Prove that $\lim_{x \to 0^+} f\left(\frac{1}{x}\right) = L$ if and only if $\lim_{x \to \infty} f(x) = L$.

5 Continuity

1. Let $f : (-1, 1) \to \mathbb{R}$ satisfy

$$|f(x)| \leq |x| \quad \forall x \in (-1, 1)$$

Show that f is continuous at $x = 0$.

2. If the function$f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = 0$ for all $x \in \mathbb{Q}$, prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

3. Give an example of two functions, one of which is not continuous at c, whose product is continuous at c.

4. Give an example of two functions, one of which is not continuous at c, whose composition is continuous at c.

5. Let $f, g : [a, b] \to \mathbb{R}$ be continuous and $f(a) < g(a)$, $f(b) > g(b)$. Show that there is $c \in (a, b)$ such that $f(c) = g(c)$.

6. Prove that the equation $\cos x = x$ has a solution in $(0, \pi/2)$.

7. Determine a real interval of length 1 where the equation $x^4 - 2x^3 - 5 = 0$ has a solution.

8. prove that it $f : [a, b] \to \mathbb{R}$ is continuous and has a maximum at $c \in (a, b)$ then f is not injective.

9. Determine which functions are uniformly continuous:

   (a) $f(x) = \sqrt[3]{x}, \quad x \in \mathbb{R}$

   (b) $f(x) = \dfrac{x^2}{x+1}, \quad x \in (0, \infty)$

   (c) $f(x) = x \sin x, \quad x \in \mathbb{R}$

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6 Differentiation

6.1 The Derivative

1. Use the definition to find the derivative of $f(x) = \dfrac{1}{x}$, $x \neq 0$

2. Find the set of points where the function f is not differentiable

   (a) $f(x) = |x^2 - 1|$

   (b) $f(x) = x|x|$

3. If $g(0) = g'(0) = 0$, find $f'(0)$ where

$$f(x) = \begin{cases} g(x) \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

4. Let

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c \end{cases}$$

   Prove that f is differentiable at $x = 0$, and evaluate $f'(0)$.

5. If the function f satisfies $|f(x)| \leq |x|^r$, where $r > 1$, prove that f is differentiable at $x = 0$.

6. Let $f : \mathbb{R} \to \mathbb{R}$. The function f is even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$, and odd if $f(-x) = -f(x)$ for all $x \in R$. If f is differentiable, prove that f' is odd when f is even, and even when f is odd.

6.2 Mean Value Theorem

7. Use the definition to show that $f(x) = \sqrt{x^2+1}$ is differentiable on $\mathbb{R}$, then prove that there is $c \in (0,1)$ such that

$$\sqrt{2} - 1 = \frac{c}{\sqrt{c^2+1}}$$

8. If $f : \mathbb{R} \to \mathbb{R}$, and there is a real constant $K > 0$ such that

$$|f(x) - f(y)| \leq K|x-y|^2 \quad \forall x, y \in \mathbb{R}$$

show that f is constant.

9. Prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

10. Prove that

$$\sqrt{1+x} < 1 + \frac{x}{2} \quad \forall\ x > 0$$

6.3 L'Hopital's Rule

11. Evaluate the following limits

(a) $\lim_{x \to 0} \frac{\cos x - 1}{x^2}$

(b) $\lim_{x \to 0^+} \left(1 + \frac{2}{x}\right)^x$

12. Let $g \in C^2(\mathbb{R})$ such that $g(0) = g'(0) = 0$ and $g''(0) = 6$. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and defined by

$$f(x) = \frac{g(x)}{x}, \qquad x \neq 0$$

Find $f(0)$, and discuss the differentiability of f at $x = 0$.

13. If $f(x) = x^2 \sin(\frac{1}{x})$, $g(x) = \sin x$, show that $\lim_{x \to 0} \frac{f(x)}{g(x)}$ exists, while $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ does not exist.

6.4 Taylor's Theorem

14. Prove that for $x > 0$,

$$1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{x+1} \leq 1 + \frac{x}{2}$$

15. Decide whether $f(0)$ is and extremum value of $f(x) = \sin x - x + \frac{x^3}{6}$.