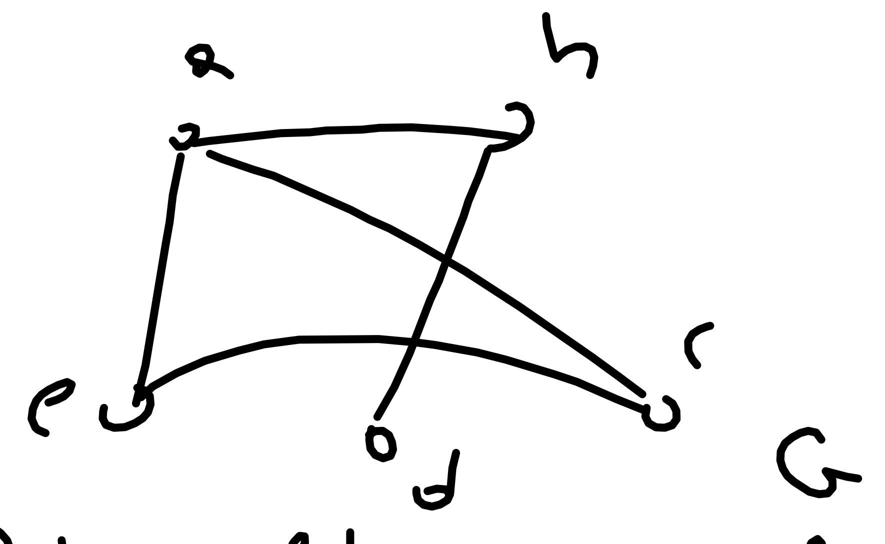
# Exercises of graphs

Exercise 9.1

3.

$V = \{a, b, c, d, e\}$



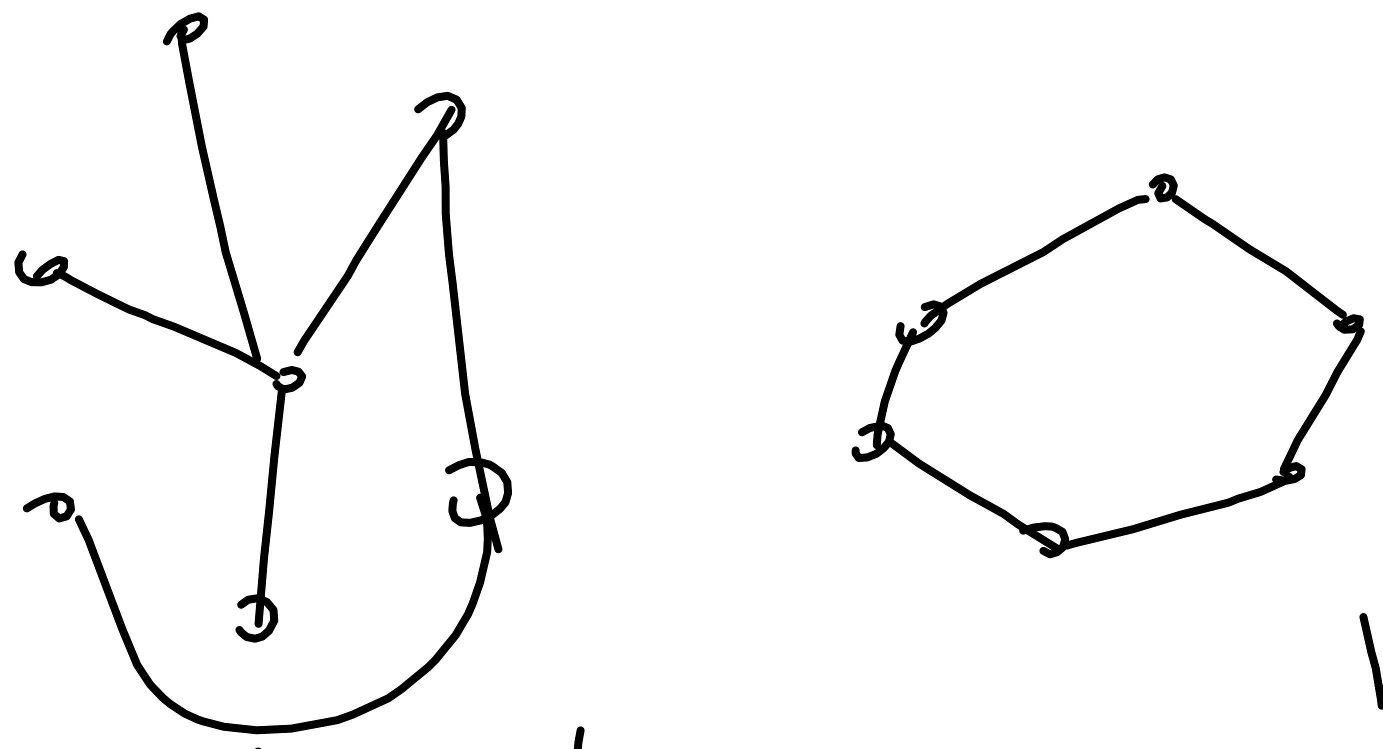

Let $R$ be a relation on $V$, defined by

$\forall x, y \in V$: $xRy \iff xy \in E$ ($xy \in E$ = $x$ and $y$ are adjacents.

In this graph: $aRb$ and $bRd$, but $a \not R d$

Thus the adjacency relation is not transitive.

6.



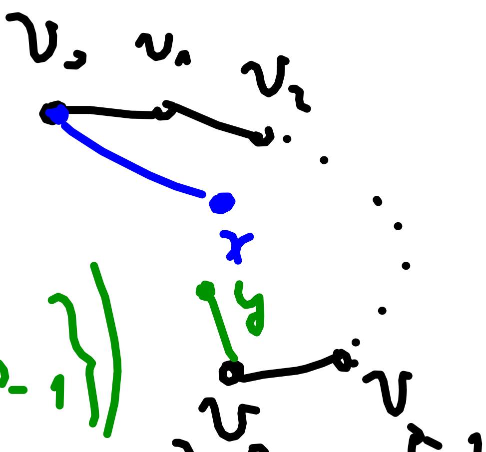
G: is connected graph and doesn't has a cycles.

( Then does not has a cycles with 4 vertices).

**9.2:** 1.] Let $G=(V,E)$ be a graph.
Suppose That $\delta(G)\geq 2$; $\left(\delta(G)=\min\{\deg x : x\in V\}\right)$
$(\delta(G)\geq 2 \iff \forall x\in V;\ \deg x\geq 2)$

**Proof:** Let $G=(V,E)$ be a graph.
Suppose That $\delta(G)\geq 2$, $(\forall x\in V;\ \deg x\geq 2)$.
Case 1: Suppose That $G$ is a connected graph.
Let $P: v_0, v_1, \dots, v_p$ is a longest path on $G$.
($P$ is exist because $G$ is Connected). $(\ell(P)=p)$

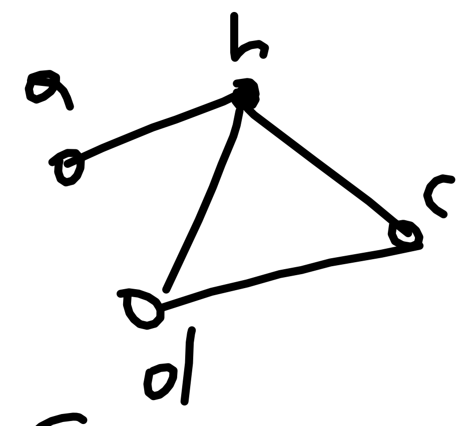
Since $\deg v_0\geq 2$ (resp. $\deg v_p\geq 2$)
Then $\exists x\in V\setminus\{v_0,v_1\}$ such That



$x v_0 \in E$ (resp. $y\in V\setminus\{v_p, v_{p-1}\}$)
• If $x\in\{v_2,\dots,v_p\}$, Then $\exists i\in\{2,\dots,p\}$, such that $x=v_i$
$C: x, v_0, v_1, \dots, v_{i-1}, v_i=x$ is a cycle, then $G$ has a cycle.
• If $x\notin\{v_0, v_1, v_2, \dots, v_p\}$, Then $P+x: x, v_0, v_1, \dots, v_p$
is a path where $\ell(P+x)=p+1 > \ell(P)=p$
Contradiction with $P$ is a longest path on $G$.
Case 2: G is not connected graph.
$G=G_1\cup\dots\cup G_k$; $k$: Connected components of $G$
$G_i$: is connected subgraph of $G$, $\forall x\in V_i$; $\deg_{G_i} x=\deg_G x\geq 2$
then $\delta(G_i)\geq 2$.
Then There is a cycle $C_i$ on $V_i$ (by case 1).
Then There are at least $k$ cycles in $G$. $(k\geq 2)$.

**Remark:** (If $G$ contains a cycle, Then $\delta(G)\geq 2$)

Deg $G=(1,2,2,3)$.

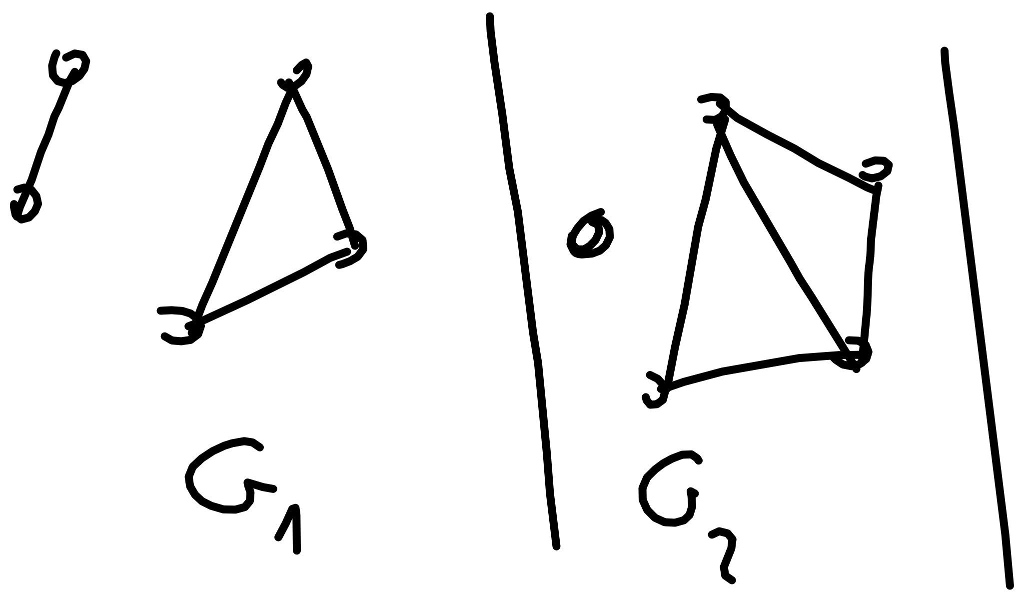



G

$C: b, c, d, b$ is a cycle
and $\delta(G)=1\ngeq 2$

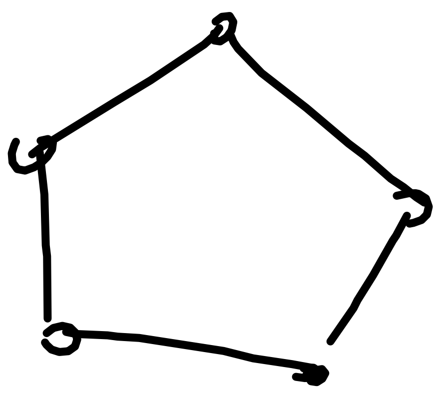
$\delta$: delta (small)
$\Delta$: delta (Capital)

2.) let $G = (V, E)$; $|V| = 5$
and two connected components of $G$.



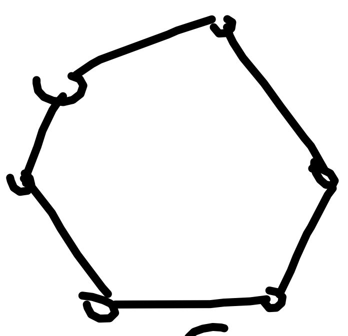
3.) let $G=(V,E)$, $|V|=5$

$\forall x\in V$; $\deg(x)=2$.

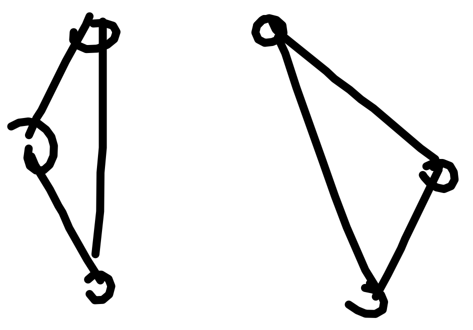


$C_5$

**Remark**: let $G=(V,E)$; $|V|=6$ and $\forall x\in V$, $\deg x=2$.



$C_6$

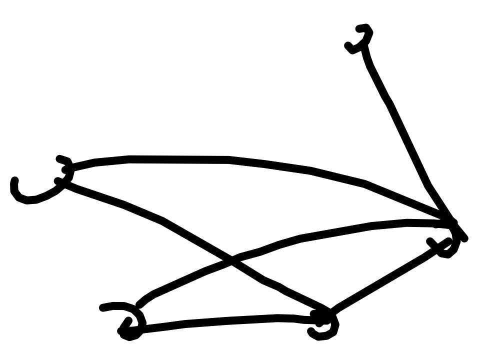


$G_2$

Exercise 9.11

3.) a) $D = \{4, 3, 2, 1\}$

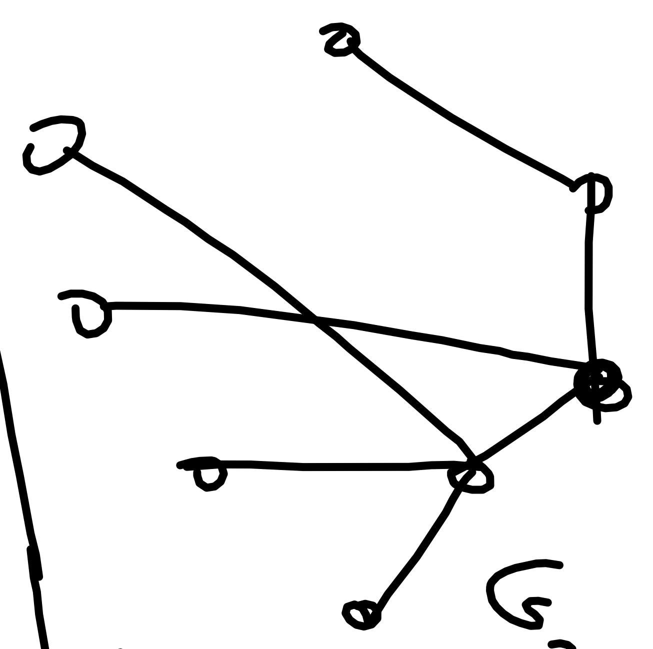
D



$G_1$

$S_1$: 1, 2, 2, 3, 4

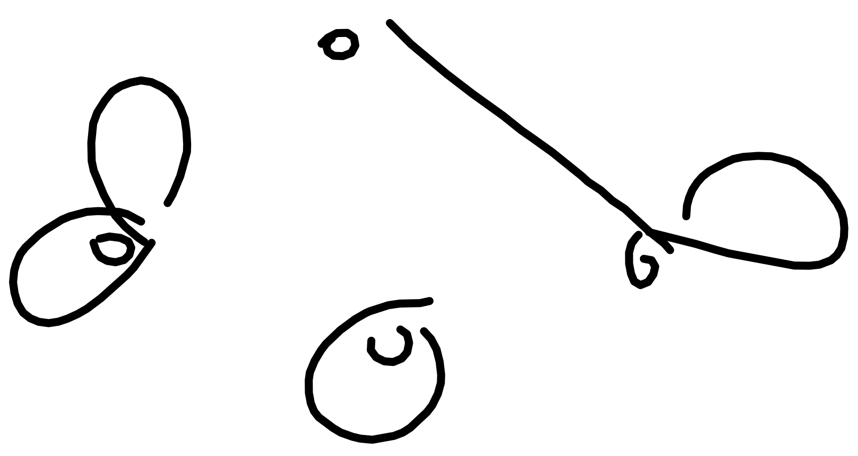
$G_1$ is a simple graph



$G_2$

$S_2$: 1, 1, 1, 1, 1, 2, 3, 4

$G_2$ is a simple graph

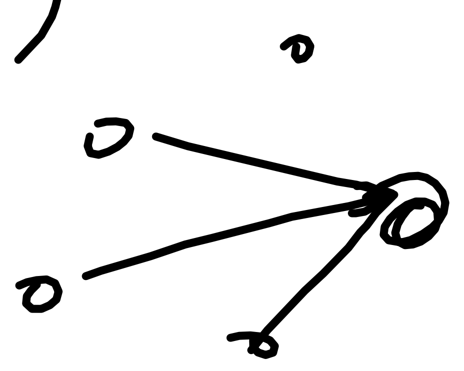


$G_3$ is not simple graph.

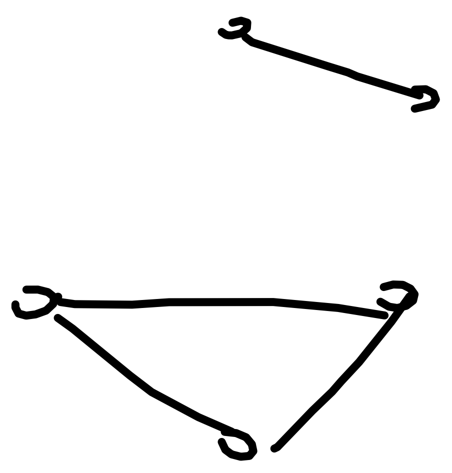
$S_3$: 1, 2, 3, 4

## Exercise 9.2

2)

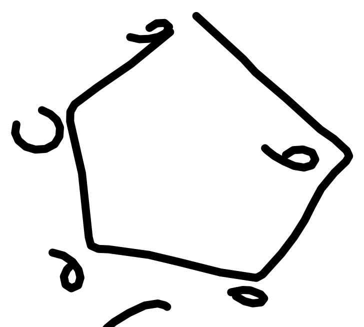


$G_1$ is with 2 components and 5 vertices



$G_2$ is a graph with 5 vertices and 2 components

3)



$C_5$

$\underline{q.4}$ : 1) let $G=(V,E)$ a solution

$\underline{n=4}$, $|E|+|\bar{E}| = \frac{n(n-1)}{2} = \frac{4(3)}{2} = 6$

$|E| = |\bar{E}|$, then $|E| = |\bar{E}| = 3$

let $V = \{v_1, v_2, v_3, v_4\}$.

$\deg v_1 + \deg v_2 + \deg v_3 + \deg v_4 = 2|E| = 6$.

G is self complementary, then G is connected

suppose that $\deg v_1 \leq \deg v_2 \leq \deg v_3 \leq \deg v_4$

$\deg v_i \neq 0$, then $\deg v_1 \geq 1$

We have $\deg v_i \leq n-1 = 4-1 = 3$ $\forall i \in \{1,..,4\}$

hence $1 \leq \deg v_1 \leq \deg v_2 \leq \deg v_3 \leq \deg v_4 \leq 3$

then the sequence degree of G is

$S = (d_1, d_2, d_3, d_4)$

with $d_1 + d_2 + d_3 + d_4 = 6$.

$d_1 \geq 1$, $d_4 \leq 3$

$\to$ $d_1 = 1$, $d_4 = 3$ $\longrightarrow$ $1 + d_2 + d_3 + 3 = 6$

$d_2 + d_3 = 2$

$d_2 = 1 = d_3$

$S_1 = (1,1,1,3) \rightsquigarrow \bar{S}_1 = (3-1, 3-1, 3-1, 3-3)$

$= (2,2,2,0)$

$\bar{G}$ has an isolated vertex, then $\bar{G}$ is disconnected contradicts with

$\cancel{G}$ $G$ is s.c, then G and $\bar{G}$ are connected

$\to$ $d_1 = 1$, $d_4 = 2$

then $1 + d_2 + d_3 + 2 = 6 \Rightarrow d_2 + d_3 = 3$

$1 = d_1 \leq d_2 \leq d_3 \leq d_4 = 2$

Then $d_2 = 1$, $d_3 = 2$

$S_2 = (1,1,2,2)$ and G is connected

$\bar{S}_2 = (3-1, 3-1, 3-2, 3-2) = (2,2,1,1) \simeq S_2$

Then $G \simeq P_4$.

$\to$ $d_1 = d_4 = 1$, then $d_2 = d_3 = 1$

hence $d_1 + d_2 + d_3 + d_4 = 4 \neq 6$

(rejected)

$\to$ $d_1 = 2$, $d_4 = 3$ (rejected)

hence $d_1 + d_2 + d_3 + d_4 \geq 9$

$\to$ $d_1 = 2$, $d_4 = 2$, then $d_1 = d_2 = d_3 = d_4 = 2$.

hence $\sum_{i=1}^{4} d_i = 8 \neq 6$ (rejected)

The only graph self complementary with 4 vertices is $P_4$.

Exercise: Let $G=(V,E)$ be a graph

$(V,\bar{E})=\bar{G}$ is a complement graph of $G$, $|V|=n$

- $G \simeq \bar{G}$ ($G$ is a self complementary graph)

Then $|E|=|\bar{E}|$

and we have $|E|+|\bar{E}|=\frac{n(n-1)}{2}$

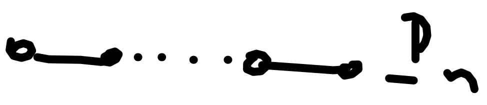
hence $2|E|=\frac{n(n-1)}{2}$, Therefore $|E|=|\bar{E}|=\frac{n(n-1)}{4}$

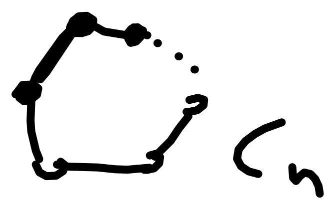
- $\frac{n(n-1)}{4}=|E|\in \mathbb{N}\cup\{0\}$,

$\Rightarrow 4/n$ or $4/(n-1) \Rightarrow \exists k\in\mathbb{N}\cup\{0\}$; $n=4k$ or $n=4k+1$

• Let $P_n$, the path of $n$ vertices ($n \in \mathbb{N}$)

$C_n$, the $n$-cycle; $n \geq 3$.





**Exercise 2**

1) $P_n$ is self complementary $\iff n=1$ or $n=4$

2) $C_n$ is self complementary $\iff n=5$.

**Exercise 1** 2) Find the number of edges in $\overline{P_n}$

3) ——— in $\overline{C_n}$

1) ——— in $\overline{K_{p,q}}$

**Solution**: Ex 1

1) $K_{p,q}=(V,E)$; $|V|=p+q$ and $|E|=p\cdot q$

$\overline{K_{p,q}}=(V,\bar{E})$, $|\bar{E}|=?$

$$|E|+|\bar{E}|=\frac{n(n-1)}{2}=\frac{(p+q)(p+q-1)}{2}$$

$$\bar{E}=\frac{(p+q)(p+q-1)}{2}-|E|=\frac{(p+q)(p+q-1)}{2}-p\cdot q$$

**Remark**

$=$

$$|\bar{E}|=\frac{p(p-1)}{2}+\frac{q(q-1)}{2}$$

2) $P_n=(V,E)$; $\overline{P_n}=(V,\bar{E})$, $|V|=n$

$|E|=n-1$, $n\in\mathbb{N}$, $|\bar{E}|=?$

We have $|E|+|\bar{E}|=\frac{n(n-1)}{2}$

$$|\bar{E}|=\frac{n(n-1)}{2}-(n-1)=\frac{(n-1)(n-2)}{2}$$

3) $C_n=(V,E)$, $|V|=n$, $|E|=n\geq 3$, $|\bar{E}|=?$

$$|E|+|\bar{E}|=\frac{n(n-1)}{2}\longrightarrow |\bar{E}|=\frac{n(n-1)}{2}-n$$

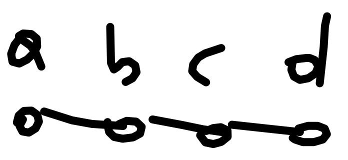
hence $|\bar{E}|=\frac{n}{2}(n-1-2)=\frac{n(n-3)}{2}$. $(n\geq 3)$

## Solution of Exercise 2

1) $P_n$ self complementary $\Leftrightarrow$ $n=1$ or $n=4$

**Proof:** "$\Leftarrow$" $n=1$: • $P_1$, • $\bar{P}_1$ $\Rightarrow \bar{P}_1 \simeq P_1$

$n=4$:




$P_4$


$\bar{P}_4$

"$\Rightarrow$": suppose that $P_n$ is a self complementary.

$\bar{P}_4$:


$P_4 \simeq \bar{P}_4$

Then $|\bar{E}| = |E| = \frac{n(n-1)}{4}$

Since $|\bar{E}| = \frac{n(n-1)}{2} - |E| = \frac{n(n-1)}{2} - (n-1) = \frac{(n-1)(n-2)}{2}$

$|\bar{E}| = \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{4}$

hence $\frac{(n-1)(n-2)}{2} - \frac{n(n-1)}{4} = 0$

$\Leftrightarrow \frac{(n-1)}{2}\left((n-2) - \frac{n}{2}\right) = 0$

$\Leftrightarrow n-1=0$ or $n-2-\frac{n}{2}=0 \Leftrightarrow n=1$ or $n=4$

**Conclusion:** $P_n$ is self complementary $\Leftrightarrow n=1$ or $n=4$

2) "$\Rightarrow$" Suppose that $C_n$ is self complementary, $C_n=(V,E)$, $n\geq 3$

Then $|\bar{E}| = |E| = \frac{n(n-1)}{4}$

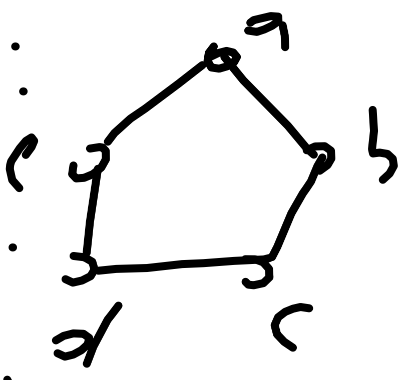
and $|\bar{E}| = \frac{n(n-1)}{2} - |E| = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$

hence $\frac{n(n-3)}{2} = \frac{n(n-1)}{4} \Leftrightarrow 2n(n-3) - n(n-1) = 0$

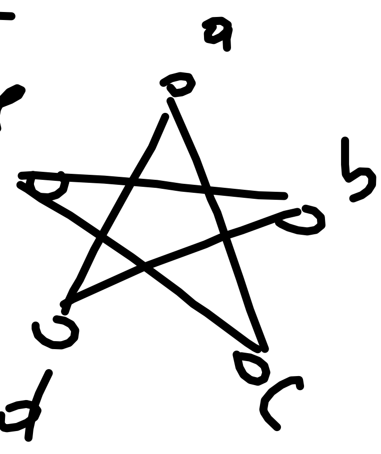
$n(2n-6-n+1) = 0 \Leftrightarrow n=0$ or $n=5$

"$\Leftarrow$": but $n\geq 3$, $n\neq 0$, then $n=5$:

let $n=5$, $C_5$:




, $\bar{C}_5$:



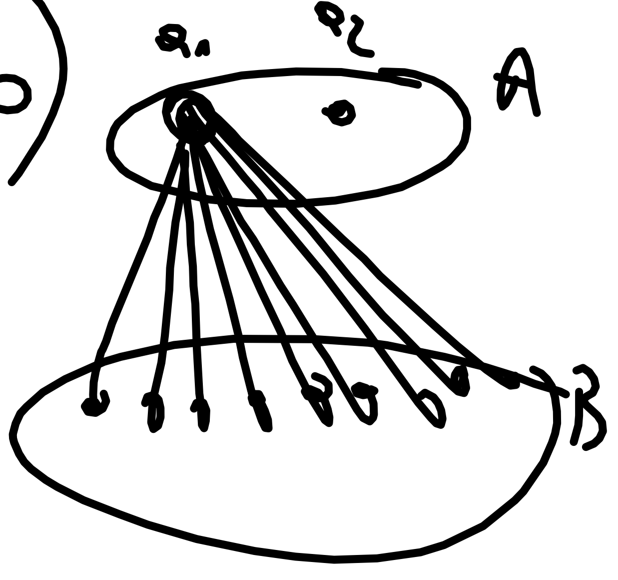

$\bar{C}_5 \simeq C_5$, then $C_5$ is self complementary.

**Conclusion:** $C_n$ is self complementary $\Leftrightarrow n=5$.

Exercise 11: 1) Suppose that there is a bipartite graph $G=(A\cup B,E)$, with the degree sequence of it is $(9,8,6,5,3,3,3,3,3,1,1)$, $|A|\leq|B|$

deg $a_1=9$, then $a_1\in A$ and $|B|\geq 9$

and $|A|\leq 2$, $(|A|\cdot|B|\neq 0)$




hence $|A|=1$ or $|A|=2$.

$\{a_3,a_4,\dots,a_{11}\}\subseteq B$

since deg $a_2=8$, then $a_2\in A$, hence $A=\{a_1,a_2\}$

and $B=\{a_3,\dots,a_{11}\}$.

Let $a_3$, deg $a_3=6$, $a_3\notin A$ and $a_3\notin B$ (deg $a_3>2$)

Conclusion: Does not exist..

Exercise 11: 2) $S: (6,6,6,6,6,5,3,3,3,3,3)$

Suppose there is a graph $G=(A\cup B, E)$ bipartite with $S$ is a sequence degree of $G$, $|A|\le|B|$

$a_1, a_2, a_3, a_4, a_5 \in A$, $\deg a_i = 6$; $1\le i\le 5$.

Then $|A|\ge 5$ and $5\le|B|\le 6$, then $5\le|A|\le 6$

$|A|=5, |B|=6$; $|A|\cdot|B| = 5\cdot 6 = 30$

$|A|=6, |B|=5$; rejected because $|A|\le|B|$.

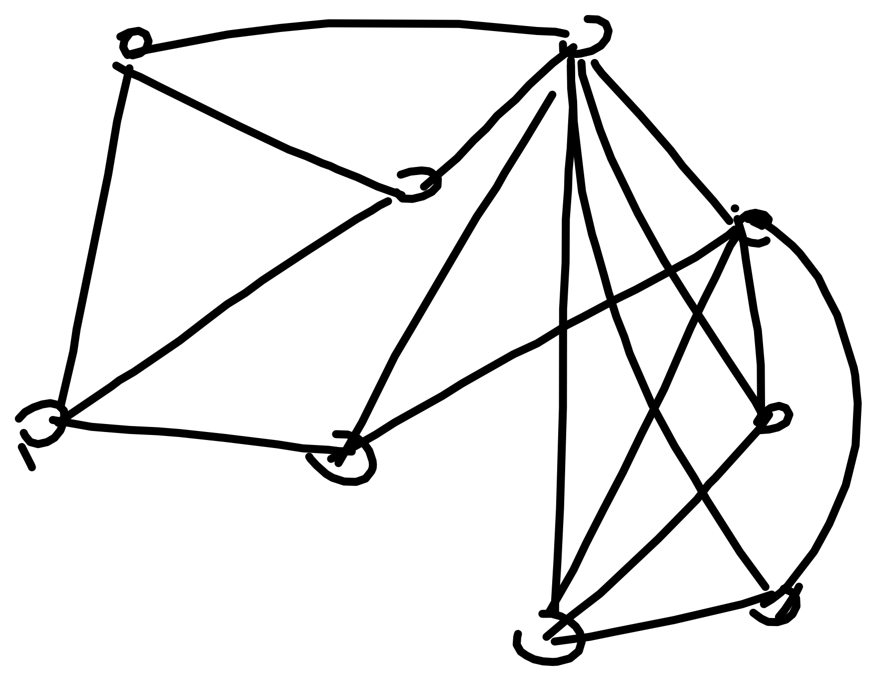
On other hand we have:

$$\sum_{x\in V}\deg x = 2|E| \iff 6\cdot 5 + 5 + 3\cdot 5 = 50$$

Then $|E| = 25 \ne 30$ (contradiction).

Then Does not exist a bipartite graph with $S$ as sequence degree.

Ex 11: 3. b) $D = \{3, 4, 5, 7\}$



Exercise $D = (1,2,3,4,4,5,6,7)$

$D_1 = (7,6,5,4,4,3,2,1)$

$D_2 = (5,4,3,3,2,0,0)$

$D_3 = (3,2,2,1,-1,0)$ is not graphic sequence

Thus D is not graphic sequence.

ii) $D = (6,6,6,6,3,3,2,2)$.

$D_1'' = (5,5,5,2,2,1,2) \rightsquigarrow D_1' = (5,5,5,2,2,2,1)$

$D_2'' = (4,4,1,1,1,1) = D_2'$

$D_3'' = (3,0,0,0,1) \Rightarrow D_3' = (3,1,0,0,0)$.

$D_4'' = (0,-1,-1,0)$ is not graphic

Then $D$ is not graphic.

Exercise 5: 2) $D = D_2 = (6,6,6,6,6,6,6,6,5,3,3,3,3,3)$

$\delta''_1 = (5,5,5,5,5,5,6,5,3,3,3,3,3)$

$\delta'_1 = (6,5,5,5,5,5,5,5,3,3,3,3,3)$

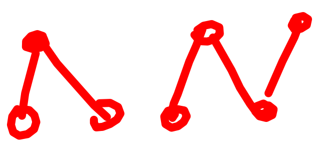
$\delta'_2 = (5,4,4,4,4,4,4,3,3,3,3,3) = \delta'$

$\delta'_3 = (4,3,3,3,3,3,3,3,3,3,3)$

$\delta'_4 = (3,3,3,3,3,3,2,2,2,2)$

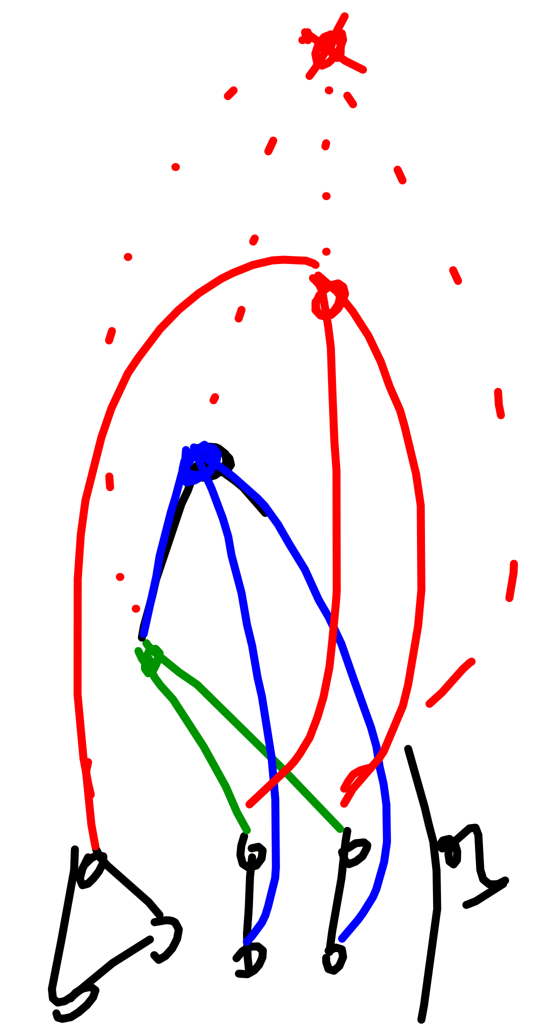
$\delta'_5 = (3,3,2,2,2,2,2,2,2)$

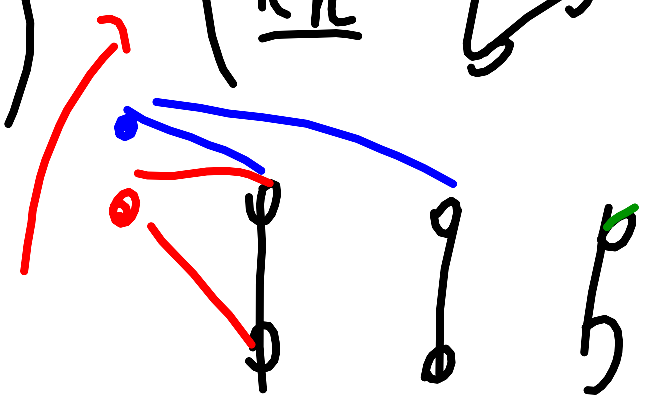
$\delta'_6 = (2,2,2,2,2,2,1,1)$



$\delta'_7 = (2,2,2,1,1,1,1)$

(Rk:





$\delta'_8 = (1,1,1,1,1,1)$

$\delta'_8$ is graphic, Then $D_2$ is graphic.