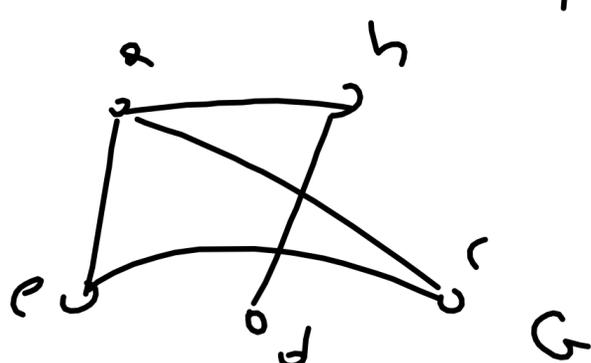


Exercises of graphs

Exercise 9.1

$$V = \{a, b, c, d, e\}$$

Ex.

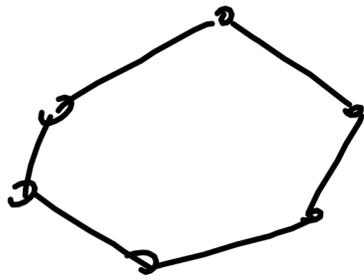
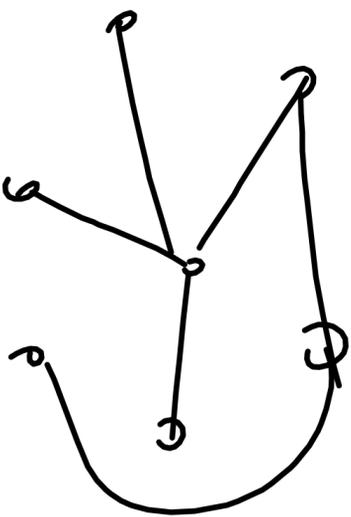


Let R be a relation on V , defined by

$$\forall x, y \in V; x R y \Leftrightarrow xy \in E \quad (xy \in E \equiv x \text{ and } y \text{ are adjacent})$$

In this graph: $a R b$ and $b R d$, but $a \not R d$.
Then the adjacency relation is not transitive.

6.



G : is connected graph and does not ^{has} a cycles.
(Then does not has a cycles with 4 vertices).

9.2: 1.) Let $G = (V, E)$ be a graph.
 Suppose that $\delta(G) \geq 2$; ($\delta(G) = \min\{\deg x : x \in V\}$)
 ($\delta(G) \geq 2 \Leftrightarrow \forall x \in V; \deg x \geq 2$)

Proof: Let $G = (V, E)$ be a graph.
 Suppose that $\delta(G) \geq 2, (\forall x \in V; \deg x \geq 2)$.

Case 1: Suppose G is a connected graph.

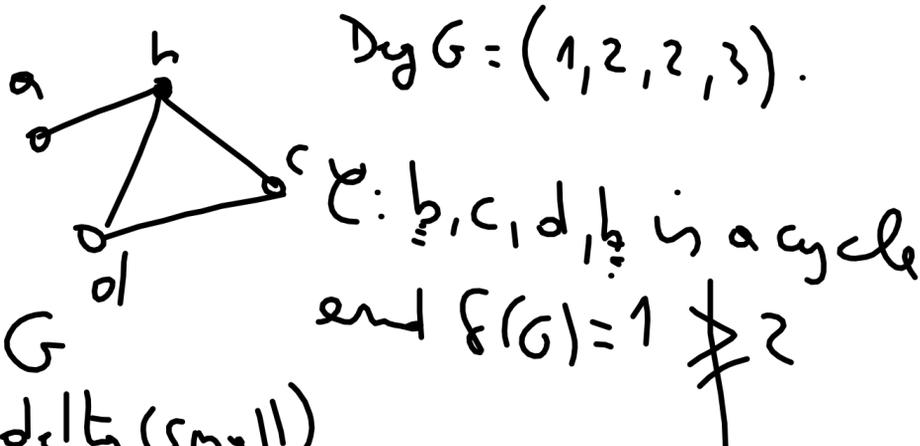
Let $P: v_0, v_1, \dots, v_p$ is a longest path on G .
 (P exist because G is connected). ($\ell(P) = p$)

Since $\deg v_0 \geq 2$ (resp. $\deg v_p \geq 2$)
 Then $\exists x \in V \setminus \{v_0, v_1\}$ such that
 $xv_0 \in E$ (resp. $y \in V \setminus \{v_p, v_{p-1}\}$)
 • If $x \in \{v_2, \dots, v_p\}$, then $\exists i \in \{2, \dots, p\}$, such that $x = v_i$
 $C: x, v_0, v_1, \dots, v_i, v_i = x$ is a cycle, then G has a cycle.
 • If $x \notin \{v_0, v_1, v_2, \dots, v_p\}$, then $P+x: x, v_0, v_1, \dots, v_p$
 is a path where $\ell(P+x) = p+1 > \ell(P) = p$
 Contradiction with P is a longest path on G .

Case 2: G is not connected graph.

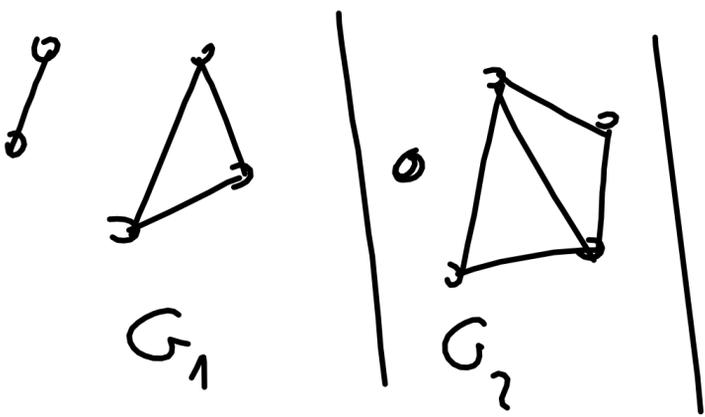
$G = G_1 \cup \dots \cup G_k$; k : Connected components of G
 G_i is connected subgraph of G ; $\forall x \in V_i; \deg x = \deg_{G_i} x \geq 2$
 Then there is a cycle C_i on V_i (by case 1).
 Then there are at least k cycles in G . ($k \geq 2$)

Remark: (If G contains a cycle, then $\delta(G) \geq 2$)



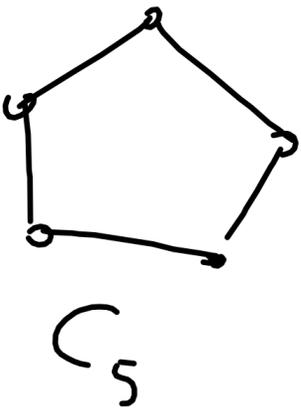
δ : delta (small)
 Δ : delta (capital)

2.) let $G = (V, E)$; $|V| = 5$
and two connected components of G .

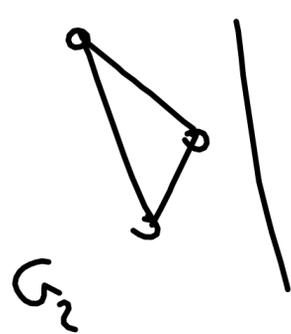
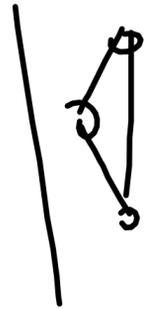
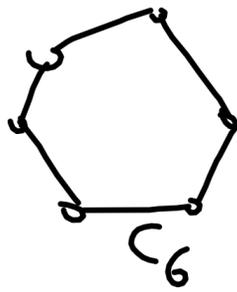


3.) Let $G = (V, E)$, $|V| = 5$

$\forall v \in V; \deg(v) = 2$.



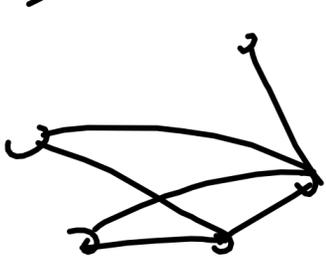
Remark: Let $G = (V, E)$; $|V| = 6$ and
 $\forall v \in V, \deg v = 2$.



Exercise 9.11

3.) a) $D = \{4, 3, 2, 1\}$

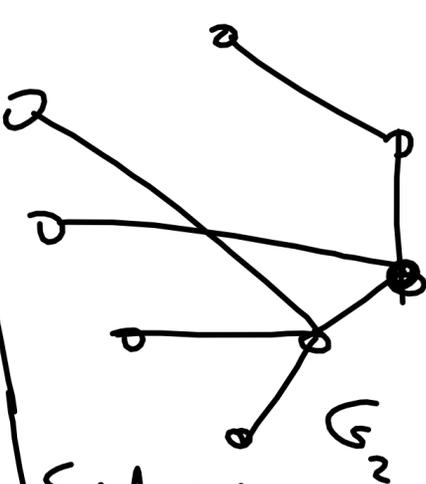
D



G_1

$S_1: 1, 2, 2, 3, 4$

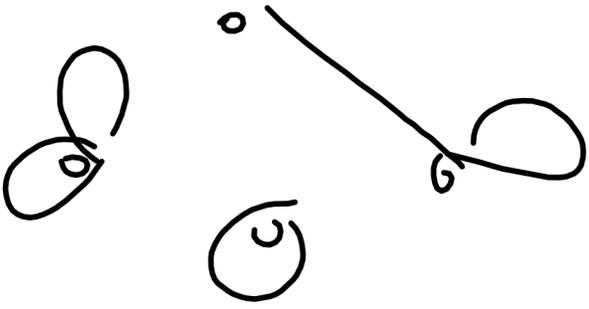
G_1 is a simple graph



G_2

$S_2: 1, 1, 1, 1, 1, 2, 3, 4$

G_2 is a simple graph

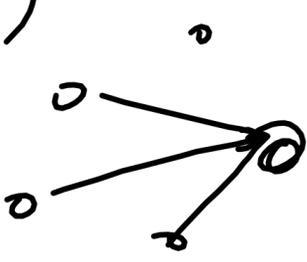


G_3 is not a simple graph.

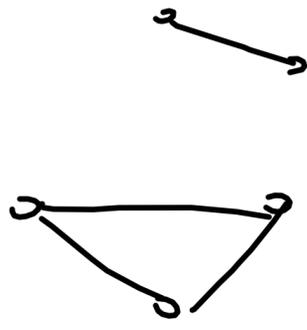
$S_3: 1, 2, 3, 4$

Exercise 9.2

2)

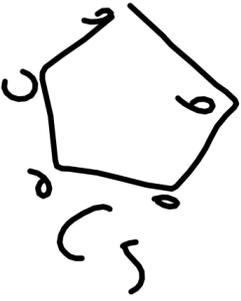


G_1 is a graph with 2 components and 5 vertices



G_2 is a graph with 5 vertices and 2 components

3)



9.4: 1) Let $G = (V, E)$ a solution
 $n = 4$, $|E| + |\bar{E}| = \frac{n(n-1)}{2} = \frac{4(3)}{2} = 6$
 $|E| = |\bar{E}|$, then $|E| = |\bar{E}| = 3$

Let $V = \{v_1, v_2, v_3, v_4\}$.
 of $v_1 + \text{deg } v_2 + \text{deg } v_3 + \text{deg } v_4 = 2|E| = 6$.
 G is self complementary, then G is connected

suppose that $\text{deg } v_1 \leq \text{deg } v_2 \leq \text{deg } v_3 \leq \text{deg } v_4$
 $\text{deg } v_i \neq 0$, then $\text{deg } v_1 \geq 1$

we have $\text{deg } v_i \leq n-1 = 3 \forall i \in \{1, \dots, 4\}$

hence $1 \leq \text{deg } v_1 \leq \text{deg } v_2 \leq \text{deg } v_3 \leq \text{deg } v_4 \leq 3$
 then the sequence degree of G is

$S = (d_1, d_2, d_3, d_4)$
 with $d_1 + d_2 + d_3 + d_4 = 6$.

$\rightarrow d_1 \geq 1, d_4 \leq 3$
 $\rightarrow d_1 = 1, d_4 = 3 \rightarrow 1 + d_2 + d_3 + 3 = 6$
 $d_2 + d_3 = 2$

$S_1 = (1, 1, 1, 3) \sim \bar{S}_1 = (3-1, 3-1, 3-1, 3-3) = (2, 2, 2, 0)$

G has an isolated vertex, then G is disconnected. Contradicts with

If G is s.c., then G and \bar{G} are connected

$\rightarrow d_1 = 1, d_4 = 2$
 then $1 + d_2 + d_3 + 2 = 6 \Rightarrow d_2 + d_3 = 3$

$1 = d_1 \leq d_2 \leq d_3 \leq d_4 = 2$

then $d_2 = 1, d_3 = 2$

$S_2 = (1, 1, 2, 2)$ and G is connected

$\bar{S}_2 = (3-1, 3-1, 3-2, 3-2) = (2, 2, 1, 1) \stackrel{?}{\sim} S_2$

then $G \cong P_4$.

$\rightarrow d_1 = d_4 = 1$, then $d_2 = d_3 = 1$

hence $d_1 + d_2 + d_3 + d_4 = 4 \neq 6$

(rejected)

$\rightarrow d_1 = 2, d_4 = 3$ (rejected)

hence $d_1 + d_2 + d_3 + d_4 \geq 9$

$\rightarrow d_1 = 2, d_4 = 2$, then $d_1 = d_2 = d_3 = d_4 = 2$.
 hence $\sum_{i=1}^4 d_i = 8 \neq 6$ (rejected)

The only graph self complementary with 4 vertices is P_4 .

Exercise: Let $G = (V, E)$ be a graph
 $(V, \bar{E}) = \bar{G}$ is a complement graph of G , $|V| = n$

• $G \cong \bar{G}$ (G is a self-complementary graph)

Then $|E| = |\bar{E}|$

and we have $|E| + |\bar{E}| = \frac{n(n-1)}{2}$

hence $2|E| = \frac{n(n-1)}{2}$, therefore $|E| = |\bar{E}| = \frac{n(n-1)}{4}$

• $\frac{n(n-1)}{4} = |E| \in \mathbb{N} \cup \{0\}$,

$\Rightarrow 4/n$ or $4/(n-1) \Rightarrow \exists k \in \mathbb{N} \cup \{0\}; n = 4k$ or $n = 4k+1$

Let P_n , the path of n vertices $\dots \dots \dots P_n$
 C_n , the n -cycle; $n \geq 3$. 

Exercise 2

- 1) P_n is self complementary $\Leftrightarrow n=1$ or $n=4$
- 2) C_n is self complementary $\Leftrightarrow n=5$.

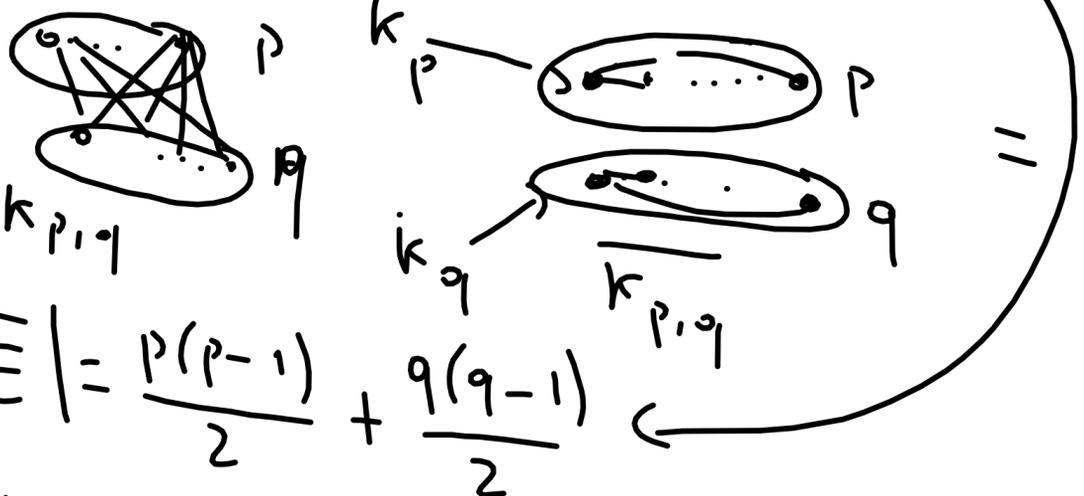
Exercise 1) Find the number of edge in $\overline{P_n}$
 3) _____
 1) _____

Solution: 1) $K_{p,q} = (V, E)$; $|V| = p+q$ and $|E| = p \cdot q$
 $K_{p,q} = (V, \overline{E})$, $|\overline{E}| = ?$

$$|E| + |\overline{E}| = \frac{n(n-1)}{2} = \frac{(p+q)(p+q-1)}{2}$$

$$|\overline{E}| = \frac{(p+q)(p+q-1)}{2} - |E| = \frac{(p+q)(p+q-1)}{2} - p \cdot q$$

Remark



$$|\overline{E}| = \frac{p(p-1)}{2} + \frac{q(q-1)}{2}$$

2) $P_n = (V, E)$; $\overline{P_n} = (V, \overline{E})$, $|V| = n$
 $|E| = n-1$, $n \in \mathbb{N}$, $|\overline{E}| = ?$

We have $|E| + |\overline{E}| = \frac{n(n-1)}{2}$

$$|\overline{E}| = \frac{n(n-1)}{2} - (n-1) = \frac{(n-1)(n-2)}{2}$$

3) $C_n = (V, E)$, $|V| = n$, $|E| = n \geq 3$, $|\overline{E}| = ?$

$$|E| + |\overline{E}| = \frac{n(n-1)}{2} \rightarrow |\overline{E}| = \frac{n(n-1)}{2} - n$$

$$\text{hence } |\overline{E}| = \frac{n}{2}(n-1-2) = \frac{n(n-3)}{2}. \quad (n \geq 3)$$

Solution of Exercise 2

1) P_n is self-complementary $\Leftrightarrow n=1$ or $n=4$
Proof: " \Leftarrow " $n=1$; $P_1 \cong \bar{P}_1 \cong P_1$
 $n=4$: $a b c d$ P_4 \bar{P}_4 $\Rightarrow \bar{P}_4 \cong P_4$

" \Rightarrow " Suppose that P_n is self-complementary
 Then $|\bar{E}| = |E| = \frac{n(n-1)}{2}$

Since $|\bar{E}| = \frac{n(n-1)}{2} - |E| = \frac{n(n-1)}{2} - \frac{n(n-1)}{2} = \frac{(n-1)(n-2)}{2}$

hence $\frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2} = 0$

$\Rightarrow \frac{(n-1)}{2} \left((n-2) - \frac{n}{2} \right) = 0$

$\Rightarrow n-1=0$ or $n-2 - \frac{n}{2} = 0 \Rightarrow n=1$ or $n=4$

Conclusion P_n is self-complementary $\Leftrightarrow n=1$ or $n=4$

2) " \Rightarrow " Suppose that C_n is self-complementary, $C_n = (V, E)$ $n \geq 3$

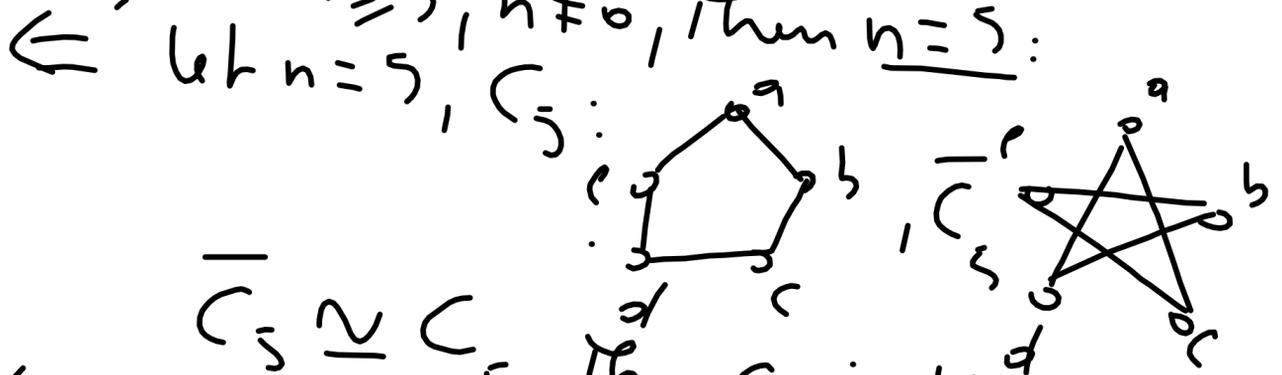
Then $|\bar{E}| = |E| = \frac{n(n-1)}{2}$

and $|\bar{E}| = \frac{n(n-1)}{2} - |E| = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$

hence $\frac{n(n-3)}{2} = \frac{n(n-1)}{2} \Rightarrow 2n(n-3) - n(n-1) = 0$

$n(2n-6-n+1) = 0 \Rightarrow n=0$ or $n=5$

" but $n \geq 3, n \neq 0$, then $n=5$.



$C_5 \cong \bar{C}_5$, then C_5 is self-complementary.

Conclusion C_n is self-complementary $\Leftrightarrow n=5$.