1-3 Classification of Differential Equations

In each of Problems 1 through 4, determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = \sin(t)$

2. $(1 + y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$

3. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 1$

4. $\dfrac{d^2y}{dt^2} + \sin(t + y) = \sin(t)$

In each of Problems 5 through 10, verify that each given function is a solution of the differential equation.

5. $y'' - y = 0; \quad y_1(t) = e^t, \quad y_2(t) = \cosh(t)$

6. $y'' + 2y' - 3y = 0; \quad y_1(t) = e^{-3t}, \quad y_2(t) = e^t$

7. $ty' - y = t^2; \quad y = 3t + t^2$

8. $y''' + 4y'' + 3y = t; \quad y_1(t) = t/3, \quad y_2(t) = e^{-t} + t/3$

9. $t^2y'' + 5ty' + 4y = 0, \quad t > 0; \quad y_1(t) = t^{-2}, \quad y_2(t) = t^{-2}\ln(t)$

2.1 Linear Differential Equations; Method of Integrating Factors

In each of the following questions

a. Draw a direction field for the given differential equation.

b. Based on an inspection of the direction field, describe how solutions behave for large t.

c. Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

1. $y' + 3y = t + e^{-2t}$
2. $y' + y = te^{-t} + 1$
3. $y' + \frac{1}{t}y = 3\cos(2t), \quad t > 0$
4. $ty' - y = t^2 e^{-t}, \quad t > 0$
5. $y' + y = 5\sin(2t)$

Find the solution of the given initial value problems

6. $y' - y = 2te^{2t}, \quad y(0) = 1$
7. $y' + 2y = te^{-2t}, \quad y(1) = 0$
8. $y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}, \quad y(\pi) = 0, \quad t > 0$
9. $ty' + (t+1)y = t, \quad y(\ln 2) = 1, \quad t > 0$

2.2 Separable Differential Equations

In each of the following questions, solve the given differential equation.

1. $y' + y^2 \sin(x) = 0$
2. $y' = \cos^2(x)\cos^2(2y)$
3. $xy' = (1 - y^2)^{1/2}$
4. $\frac{dy}{dx} = \frac{-x}{y}$

In each of the following questions

a. Find the solution of the given initial value problem in explicit form.

b. Plot the graph of the solution.

c. Determine (at least approximately) the interval in which the solution is defined.

5. $y' = (1 - 2x)/y$, $y(1) = -2$
6. $xdx + ye^{-x}dy = 0$, $y(0) = 1$
7. $y' = xy^3(1 + x^2)^{-1/2}$, $y(0) = 1$
8. $y' = 2x/(1 + 2y)$, $y(2) = 0$
9. Solve the initial value problem

$$y' = \frac{3x^2}{3y^2 - 4}, \quad y(1) = 0$$

and determine the interval in which the solution is valid.

Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

10. Solve the equation
$$\frac{dy}{dx} = \frac{ay+b}{cy+d},$$
where $a, b, c,$ and d are constants.

11. Solve the equation
$$\frac{dy}{dx} = \frac{y-4x}{x-y},$$
where $a, b, c,$ and d are constants.

12. Solve the following equations
    (a) $ydx = 2(x+y)dy$
    (b) $(y^2 + yx)dx = x^2 dy$
    (c) $\dfrac{dy}{dx} = \dfrac{x+3y}{3x+y}$
    (d) $(x^2 + 2y^2)dx = xydy$, $y(-1) = 1$.

2.4 Difference Between Linear and Non Linear Differential equations

Exercise 1 :
Solve the following equations:

1. $xy' + y = \dfrac{1}{y^2}$;
2. $xy' - (1+x)y = xy^2$;
3. $y^{\frac{1}{2}}y' + y^{\frac{3}{2}} = 1, \quad y(0) = 4.$

Exercises on Chapter 3

3.1 Homogeneous Equations with Constant Coefficients

Exercise 1 :
Find the general solution of the given differential equation.

1. $y'' + 2y' - 3y = 0$
2. $6y'' - y' - y = 0$
3. $4y'' - 9y = 0$

Exercise 2 :
Find the solution of the given initial value problem.

1. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$
2. $2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
3. $y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0$

Exercise 3 :
Find a differential equation whose general solution is $y = c_1 e^{x^2} + c_2 e^{-x^2}$.

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

Exercise 4 :
Find the Wronskian of the given pair of functions.

1. $\cos(t), \quad \sin(t)$

2. $e^{-2t}, \quad te^{-2t}$

3. $e^t \sin(t), \quad e^t \cos(t)$

Exercise 5 :
Determine the longest interval in which the given initial value problem is certain to have a unique twice-differentiable solution. Do not attempt to find the solution.

1. $t(t-4)y'' + 3ty' + 4y = 2, \quad y(3) = 0, \quad y'(3) = -1$

2. $y'' + (\cos(t))y' + 3(\ln|t|)y = 0, \quad y(2) = 3, \quad y'(2) = 1$

Exercise 6 :

1. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation $t^2y'' - 2y = 0$ for $t > 0$. Then show that $y = c_1t^2 + c_2t^{-1}$ is also a solution of this equation for any c_1 and c_2.

2. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

Exercise 7 :
Find the fundamental set of solutions specified by Theorem 3.2.5 for the given differential equation and initial point. $y'' + 4y' + 3y = 0, \quad t_0 = 1$

Exercise 8 :
Verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions? $x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0; \quad y_1(x) = x, \quad y_2(x) = xe^x$

3.3 Complex Roots of the Characteristic Equation

Exercise 9 :
Find the general solution of the given differential equation.

1. $y'' - 2y' + 2y = 0$
2. $y'' + 2y' + 2y = 0$
3. $9y'' + 9y' - 4y = 0.$

Exercise 10 :
Find the solution of the given initial value problem.

1. $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$
2. $y'' + y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -4.$

Exercise 11 :
Use the method of Problem 25 to solve the given equation for $t > 0$.

1. $t^2y'' + ty' + y = 0$
2. $t^2y'' - 4ty' - 6y = 0$
3. $t^2y'' + 7ty' + 10y = 0$

Exercise 12 :
Determine the general solution of the given differential equation that is valid in any interval not including the singular point.

1. $x^2y'' + 4xy' + 2y = 0$
2. $(x+1)^2y'' + 3(x+1)y' + 0.75y = 0$
3. $x^2y'' - 3xy' + 4y = 0$
4. $x^2y'' - xy' + y = 0$
5. $x^2y'' + 6xy' - y = 0$
6. $2x^2y'' - 4xy' + 6y = 0$
7. $x^2y'' - 5xy' + 9y = 0$

8. $(x-2)^2y'' + 5(x-2)y' + 8y = 0$

Exercise 13 :
Find the solution of the given initial-value problem.

$$4x^2y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3.$$

3.4 Repeated Roots; Reduction of Order

Exercise 14 :
Find the general solution of the given differential equation.

1. $y'' - 2y' + y = 0$
2. $9y'' + 6y' + y = 0$
3. $y'' - 6y' + 9y = 0$
4. $16y'' + 24y' + 9y = 0$

Exercise 15 :
Solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

1. $9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$
2. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1$

Exercise 16 :
Use the method of reduction of order to find a second solution of the given differential equation.

1. $t^2y'' + 2ty' - 2y = 0$, $t > 0$; $y_1(t) = t$
2. $xy'' - y' + 4x^3y = 0$, $x > 0$; $y_1(x) = \sin(x^2)$

3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

Exercise 17 :
Find the general solution of the given differential equation.

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' - y' - 2y = -2t + 4t^2$
3. $y'' + 2y' + y = 2e^{-t}$

Exercise 18 :
Find the solution of the given initial value problem.

1. $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$
2. $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$
3. $y'' + 2y' + 5y = 4e^{-t}\cos(2t), \quad y(0) = 1, \quad y'(0) = 0$

3.6 Variation of Parameters

Exercise 19 :
Use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients. $y'' - y' - 2y = 2e^{-t}$

Exercise 20 :
Find the general solution of the given differential equation.

1. $y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$
2. $4y'' + y = 2\sec(t/2), \quad -\pi < t < \pi$
3. $y'' - 2y' + y = e^t/(1 + t^2)$

Exercise 21 :

Verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation.

10. $t^2y'' - 2y = 3t^2 - 1, \quad t > 0; \quad y_1(t) = t^2, \quad y_2(t) = t^{-1}$
11. $x^2y'' - 3xy' + 4y = x^2\ln(x), \quad x > 0; \quad y_1(x) = x^2, \quad y_2(x) = x^2\ln(x)$
12. $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 3x^{3/2}\sin(x), \quad x > 0; \quad y_1(x) = x^{-1/2}\sin(x), \quad y_2(x) = x^{-1/2}\cos(x)$

Exercises on Chapter 4

4-1 General Theory of n^{th} Order Linear Differential Equations

Exercise 1 :
Determine whether the given functions are linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

1. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t$
2. $f_1(t) = 2t - 3, \quad f_2(t) = 2t^2 + 1, \quad f_3(t) = 3t^2 + t$
3. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$

Exercise 2 :
Verify that the given functions are solutions of the differential equation, and determine their Wronskian.

8. $y^{(4)} + y'' = 0; \quad 1, \quad t, \quad \cos(t), \quad \sin(t)$
9. $y''' + 2y'' - y' - 2y = 0; \quad e^t, \quad e^{-t}, \quad e^{-2t}$
10. $x^3y''' + x^2y'' - 2xy' + 2y = 0; \quad x, \quad x^2, \quad 1/x$

4.2 Homogeneous Differential Equations with Constant Coefficients

Exercise 3 :
Find the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 0$

2. $y''' - 3y'' + 3y' - y = 0$

3. $y^{(4)} - 4y''' + 4y'' = 0$

4. $y^{(4)} + 2y'' + y = 0$

5. $y''' + 5y'' + 6y' + 2y = 0$

Exercise 4 :
Verify that $y(t) = 3e^{-t} + \frac{1}{2}\cos(t) - \sin(t)$ is the solution to $y^{(4)} - y = 0, y(0) = \frac{7}{2}, y'(0) = -4, y''(0) = \frac{5}{2}, y'''(0) = -2$.

4-3 The Method of Undetermined Coefficients

Exercise 5 :
Determine the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 2e^{-t} + 3$

2. $y^{(4)} - y = 3t + \cos(t)$

3. $y^{(4)} - 4y'' = t^2 + e^t$

Exercise 6 :
Find the solution of the given initial-value problem. Then plot a graph of the solution.

1. $y''' + 4y' = t$; $y(0) = y'(0) = 0$, $y''(0) = 1$

2. $y^{(4)} + 2y''' + y'' + 8y' - 12y = 12\sin(t) - e^{-t}$; $y(0) = 3$, $y'(0) = 0$, $y''(0) = -1$, $y'''(0) = 2$

1 Systems of Linear Differential Equations

Exercise 7 :
Solve the following systems

1. $\begin{cases} x' &= 4x + 7y \\ y' &= x - 2y \end{cases}$

2. $\begin{cases} x' &= -5x - y \\ y' &= 4x - y \end{cases}, \; x(1) = 0, \; y(1) = 1.$

3. $\begin{cases} x' &= 4y + 1 \\ y' &= -x + 2. \end{cases}.$

4. $\begin{cases} (D^2 + 5)x - 2y &= 0 \\ -2x + (D^2 + 2)y &= 0 \end{cases}.$

5. $\begin{cases} Dx + D^2 y &= e^{3t} \\ (D+1)x + (D-1)y &= 4e^{3t}. \end{cases}.$

6. $\begin{cases} (2D^2 - D - 1)x - (2D+1)y &= 1 \\ (D-1)x + Dy &= -1. \end{cases}.$

Exercises on Chapter 5

5.1 Review on Power Series

Exercise 1 :

Rewrite the given expression as a single power series whose generic term involves x^n.

1. $$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

2. $$x\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{k=0}^{\infty} a_k x^k$$

3. $$x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

5.2 Series Solutions Near an Ordinary Point, Part I

Exercise 2 :

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation that the coefficients must satisfy.

(b) Find the first four nonzero terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W[y_1, y_2](x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

1. $y'' - y = 0, \quad x_0 = 0$
2. $y'' - xy' - y = 0, \quad x_0 = 0$
3. $y'' + k^2x^2y = 0, \quad x_0 = 0, \quad k$ a constant
4. $y'' + xy' + 2y = 0, \quad x_0 = 0$
5. $(3 - x^2)y'' - 3xy' - y = 0, \quad x_0 = 0$
6. $2y'' + (x + 1)y' + 3y = 0, \quad x_0 = 2$

Exercise 3 :

(a) Find the first five nonzero terms in the solution of the given initial-value problem.

(b) Plot the four-term and the five-term approximations to the solution on the same axes.

(c) From the plot in part b, estimate the interval in which the four-term approximation is reasonably accurate.

1. $y'' + xy' + 2y = 0, \quad y(0) = 4, \quad y'(0) = -1;$ see Problem 7
2. $(1 - x)y'' + xy' - y = 0, \quad y(0) = -3, \quad y'(0) = 2$

6.1 Definition of the Laplace Transform

Exercise 1 :
Find the Laplace transform of each of the following functions:

1. $f(t) = t$;
2. $f(t) = t^2$;
3. $f(t) = t^n$, where n is a positive integer.

Exercise 2 :
Find the Laplace transform of $f(t) = \cos(at)$, where a is a real constant.
Recall that $\cosh(bt) = \dfrac{1}{2}(e^{bt} + e^{-bt})$ and $\sinh(bt) = \dfrac{1}{2}(e^{bt} - e^{-bt})$.

Exercise 3 :
Use the linearity of the Laplace transform to find the Laplace transform of the given function; a and b are real constants.

1. $f(t) = \cosh(bt)$;
2. $f(t) = \sinh(bt)$.

Recall that $\cos(bt) = \dfrac{1}{2}(e^{ibt} + e^{-ibt})$ and $\sin(bt) = \dfrac{1}{2}(e^{ibt} - e^{-ibt})$.

Exercise 4 :
Use the linearity of the Laplace transform to find the Laplace transform of the function $f(t) = \cosh(bt)$; b is a real constant. Assume that the necessary elementary integration formulas extend to this case.

Exercise 5 :
Use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

1. $f(t) = te^{at}$;
2. $f(t) = t\sin(at)$.

Exercise 6 :
Find the Laplace transform of the given function.

1. $f(t) = \begin{cases} 1 & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$

2. $f(t) = \begin{cases} t & t \in [0, 1) \\ 2 - t & t \in [1, 2) \\ 0 & t \in [2, \infty). \end{cases}$

6.2 Solution of Initial Value Problems

Exercise 1 :
Find the inverse Laplace transform of the given function.

1. $F(s) = \dfrac{3}{s^2 + 4}$;

2. $F(s) = \dfrac{4}{(s-1)^3}$;

3. $F(s) = \dfrac{2s + 2}{s^2 + 2s + 5}$;

4. $F(s) = \dfrac{8s^2 - 4s + 12}{s(s^2 + 4)}$;

Exercise 2 :
Use the Laplace transform to solve the given initial value problem.

1. $y'' + 3y' + 2y = 0, \quad y(0) = 1, y'(0) = 0;$

2. $y'' - 2y' + 4y = 0, \quad y(0) = 2, y'(0) = 0;$

3. $y'' + 2y' + 5y = 0, \quad y(0) = 2, y'(0) = -1;$

4. $y'' - 2y' + 2y = 0, \quad y(0) = 0, y'(0) = 1.$

6.3 Step Functions

Exercise 1 :
In each of the following question:

a Sketch the graph of the given function.

b Express $f(t)$ in terms of the unit step function $u_c(t)$.

1. $f(t) = \begin{cases} 0 & t \in [0,3) \\ -2 & t \in [3,5) \\ 2 & t \in [5,7) \\ 1 & t \geq 7. \end{cases}$

2. $f(t) = \begin{cases} t & t \in [0,2) \\ t & t \in [2,5) \\ 7-t & t \in [5,7) \\ 0 & t \geq 7. \end{cases}$

Exercise 2 :
Find the inverse Laplace transform of the given function.

1. $F(s) = \dfrac{3!}{(s-2)^4}$;

2. $F(s) = \dfrac{e^{-2s}}{s^2+s-2}$;

3. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2}$.

Exercise 3 :
Suppose that $F(s) = \mathcal{L}(f(t))$ exists for $s > a \geq 0$.

1. Show that if c is a positive constant, then

$$\mathcal{L}(f(ct)) = \frac{1}{c}F(\frac{s}{c}), \quad s > ca.$$

2. Show that if k is a positive constant, then

$$\mathcal{L}^{-1}F(ks) = \frac{1}{k}f(\frac{t}{k}).$$

3. Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}(F(as+b)) = \frac{1}{a}e^{-\frac{b}{a}t}f(\frac{t}{a}).$$

Exercise 4 :

Find the inverse Laplace transform of the following function $F(s) = \frac{2s+1}{4s^2+4s+5}$;

Exercise 5 :

Find the Laplace transform of the given function $f(t) = \begin{cases} 1 & t \in [0,1) \\ 0 & t \geq 1. \end{cases}$