$g(x)$ attains it's max at $x=\frac{n}{n+1}$

$$\Rightarrow g\left(\tfrac{n}{n+1}\right)=\left(\tfrac{n}{n+1}\right)^n\left(1-\tfrac{n}{n+1}\right)=\left(\tfrac{n}{n+1}\right)^n\left(\tfrac{n+1-n}{n+1}\right)$$

$$=\left(\tfrac{n}{n+1}\right)^n\left(\tfrac{1}{n+1}\right)$$

$$\Rightarrow \sup_{x\in[0,1]}\left|f_n(x)-0\right|=\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1}$$

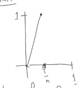
$$\leq \frac{1}{n+1}\to 0$$

So f_n converges uniformly to f on $[0,1]$.
$f\equiv 0,\ x\in[0,1]$

Ex 1.8.:

$$f_n(x)=\begin{cases} nx, & 0\leq x\leq \frac{1}{n} \\ 0, & \frac{1}{n}<x\leq 1\end{cases},\quad x\in[0,1]$$

Pointwise limit



$x\in[0,1]$

$$\lim_{n\to\infty} f_n(x)=?$$

when $x=0 \Rightarrow \forall n\in\mathbb{N},\ f_n(0)=0$

when $x>0 \Rightarrow \exists N\in\mathbb{N}$ such that
$\frac{1}{N}<x$

then for every $n\geq N$ we have

$$\frac{1}{n}\leq\frac{1}{N}<x$$

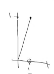
$$\Rightarrow \forall n\geq N,\ f_n(x)=0$$

$$\Rightarrow \lim_{n\to\infty} f_n(x)=0$$

So $f_n \xrightarrow{p.w} 0$. on $[0,1]$

Uniform convergence:

$$\sup_{x\in[0,1]}\left|f_n(x)-0\right|=\sup_{x\in[0,1]} f_n(x)=f_n\left(\tfrac{1}{n}\right)=1$$

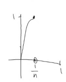


$\Rightarrow$ the convergence is not uniform

Ex 1.9:

$$f_n(x)=\begin{cases} \sqrt{nx}, & 0\leq x\leq \frac{1}{n} \\ 0, & \frac{1}{n}<x\leq 1\end{cases}$$

Pointwise limit:



when $x=0 \Rightarrow f_n(0)=0,\ \forall n\in\mathbb{N}$

when $x>0$ by Archamidean Property

$\exists N\in\mathbb{N}$ Such that $\frac{1}{N}<x$

$\forall n\geq N,\ \frac{1}{n}\leq\frac{1}{N}<x$

$$\Rightarrow \lim_{n\to\infty} f_n(x)=0,\quad \forall x\in[0,1]$$

So $f_n(x)\xrightarrow{p.w.} 0$

Uniform convergence:

$$\sup_{x\in[0,1]}\left|f_n(x)-0\right|=\sup_{x\in[0,1]} f_n(x)=f_n\left(\tfrac{1}{n}\right)=1$$

So f_n is not converging uniformly

Ex 1.10:

if $f_n \xrightarrow{u} f$, on D and $f_n \xrightarrow{u} f$, on E

then $f_n \xrightarrow{u} f$ on $D\cup E$

Sol.:

$$\sup_{x\in D\cup E}\left|f_n(x)-f(x)\right| =$$

$$=\max\left(\sup_{x\in D}|f_n-f|,\ \sup_{x\in E}|f_n-f|\right)$$

Since $\sup_{x\in D}|f_n-f|\longrightarrow 0$ and $\sup_{x\in E}|f_n-f|\longrightarrow 0$

it follows that $\sup_{x\in D\cup E}|f_n-f|\longrightarrow 0$

Ex 1.14:

$$f_n(x) = \frac{x^n}{1+x^n}, \quad x \in [0,2]$$

Pointwise limit:

When $x \in (0,1) \Rightarrow \lim_{n\to\infty} f_n(x) = 0$

when $x = 1 \Rightarrow \lim_{n\to\infty} f_n(1) = \frac{1}{2}$

when $x \in (1,2] \Rightarrow \lim_{n\to\infty} f_n(x) = 1$

$$f_n(x) = \frac{1}{1+\left(\frac{1}{x}\right)^n} \longrightarrow \frac{1}{1+0}$$

$$1 < x \Rightarrow 0 < \frac{1}{x} < 1 \Rightarrow \left(\frac{1}{x}\right)^n \longrightarrow 0$$

$f_n \xrightarrow{P.W.} f$ on $[0,2]$ where

$$f(x) = \begin{cases} 0, & x \in [0,1) \\ \frac{1}{2}, & x = 1 \\ 1, & x \in (1,2] \end{cases}$$

$f(x)$ is discontinuous at $x=1$

and for all n f_n continuous on $[0,2]$

then $f_n \overset{U}{\not\longrightarrow} f$ on $[0,2]$

Ex 1.15:

$$f_n(x) = \begin{cases} \frac{1}{n}, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases}$$

f_n is discontinuous at each Point of $\mathbb{R}$

indeed, let $x_0 \in \mathbb{R}$

Since $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{R}$

there exist two sequences (x_n) of rational numbers, and (t_n) of irrational numbers

S.T. $(x_n) \longrightarrow x_0$ and $(t_n) \longrightarrow x_0$

So if f is cont. at x_0

then $f(x_0) = \lim_{m\to\infty} f_n(x_m) = \frac{1}{n} =$ impossible

$= \lim_{m\to\infty} f_n(t_m) = 0$

$\sup_{x\in\mathbb{R}} |f_n(x)| = \frac{1}{n} \longrightarrow 0$ as $n \to \infty$

So $f_n \xrightarrow{U} f(x)$ on $\mathbb{R}$

where $f \equiv 0$ (f conti on $\mathbb{R}$)

Ex 1.16: $\varphi \in C[0,1]$

$$f_n(x) = x^n \varphi(x), \quad x \in [0,1]$$

Show that f_n converge uniformly

iff $\varphi(1) = 0$

Pointwise limit:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x \in [0,1) \\ \varphi(1), & x = 1 \end{cases} = f(x)$$

So $\begin{cases} f_n \xrightarrow{U} f(x) \\ f_n \text{ cont on } [0,1] \end{cases} \Rightarrow f$ cont at $x=1$

$\Rightarrow \varphi(1) = 0$

$\left(\varphi(1) = f(1) = \lim_{x\to 1} f(x) = 0\right)$

"$\Leftarrow$" Assume that $\varphi(1) = 0$ and show that $f_n \xrightarrow{U} 0$ on $[0,1]$

$\varepsilon > 0$

$\varphi(x)$ cont. at $x=1$, then $\exists \eta > 0$ (?)

Point

$\forall x \in (1-\eta, 1], \ |\varphi(x)| < \frac{\varepsilon}{2}$

$$\left[\begin{array}{l} \exists \eta > 0, \\ \forall x \in [0,1] \ \ |x-1| < \eta \Rightarrow |\varphi(x) - \varphi(1)| < \frac{\varepsilon}{2} \\ 1-\eta < x \leq 1 \end{array} \right]$$

On $[0, 1-\eta]$ $f_n \xrightarrow{U} 0$, then

$\exists N \in \mathbb{N}, \ \forall n \geq N \Rightarrow |f_n(x)| < \frac{\varepsilon}{2}, \ \forall x \in [0, 1-\eta]$

$n \geq N$,

$$\sup_{x\in[0,1]} |f_n(x)| \leq \max\left(\sup_{x\in[0,1-\eta]} |f_n|, \sup_{x\in[1-\eta,1]} |f_n(x)| \right) \leq \frac{\varepsilon}{2}$$

$$\leq \frac{\varepsilon}{2} < \varepsilon$$

$\Rightarrow \sup_{[0,1]} |f_n| \longrightarrow 0$ as $n \to \infty$

$$|f_n(x) - f(x)| < \varepsilon$$

$$|f(x) - f_n(x)| < \varepsilon$$

$$-\varepsilon < f(x) - f_n(x) < \varepsilon$$

$$f_n(x) - \varepsilon < f(x) < f_n(x) + \varepsilon$$

$$f_N(x) - \varepsilon < f_n(x)$$

$$L(f_N, P) - \varepsilon(b-a) = \sum_{k=1}^{n} m_k \Delta x_k - \varepsilon \sum_{k=1}^{n} \Delta x_k$$

$$= \sum_{k=1}^{n} \left(m_k^{f_N} - \varepsilon\right) \Delta x_k \leq \sum_{k=1}^{n} m_k^{f} \Delta x_k$$

$$\leq L(f, P_m)$$

$$f_n'(x) = 1 - \underbrace{\frac{\cos(nx)}{n}}_{\to 0}$$

$$f_n'(x) - 1 = -\frac{\cos(nx)}{n}$$

$$-1 \leqslant \cos(nx) \leqslant 1$$

$$1 \geqslant -\cos(nx) \geqslant -1$$

$$0 \leftarrow \frac{1}{n} \geqslant \frac{-\cos(nx)}{n} \geqslant \frac{-1}{n} \to 0$$

$$\downarrow$$

$$0$$

$$f_n(x) = \frac{x}{1+nx} \qquad \left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

$$\sup_{x\in[0,1]} \left| f_n(x) - 0 \right|$$

$$f_n'(x) = \frac{1+nx-nx}{(1+nx)^2} = \frac{1}{(1+nx)^2} > 0$$

So f_n is increasing on $[0,1]$

$$\Rightarrow f_n(1) \geqslant f_n(x)$$
$$x \in [0,1]$$

$$\sup_{x\in[0,1]} f_n(x) = \frac{1}{1+n} \to 0 \text{ as } n \to \infty$$

exercises: Uniform convergence of sequences.

Ex 1: $f_n(x) = \frac{\sin(nx)}{nx}$, $x \in (0,1)$

Point wise limit

$$\lim_{n\to\infty} \frac{\sin(nx)}{nx} = 0$$
$x \in (0,1)$

because $|\sin(nx)| \leqslant 1$

- $\frac{|\sin(nx)|}{nx} \leqslant \frac{1}{nx}$, $\forall n \in \mathbb{N}$
- $\lim_{n\to\infty} \frac{1}{nx} = 0$

by squeeze theorem, $f_n \xrightarrow{p.w} 0$, on $(0,1)$

Uniform convergence

on $(0,1)$

take $x_n = \frac{\pi}{2n}$,

$f_n(x_n) = \frac{2\sin(\frac{\pi}{2})}{\pi} = \frac{2}{\pi} \not\to 0$, as $n \to \infty$

$\left(\sup_{x\in(0,1)} |f_n(x)| \geqslant \sup_n |f(x_n)| \geqslant \frac{2}{\pi} : \forall n \in \mathbb{N} \right)$

$$f_n \overset{U}{\not\longrightarrow} 0, \text{ on } (0,1)$$

On $[a,1]$ $(0 < a \leqslant 1)$

$$|f_n(x)| = \left| \frac{\sin(nx)}{nx} \right| \leqslant \frac{1}{n|x|} \leqslant \frac{1}{na} \quad \forall x \in [a,1]$$

$$\sup_{x\in[a,1]} |f_n(x)| \leqslant \frac{1}{na} \to 0$$

$$\sup_{x\in[a,1]} |f_n(x)| \longrightarrow 0 \text{ as } n \to \infty$$

$$\Rightarrow f_n \xrightarrow{U} 0 \text{ on } [a,1]$$

Ex 2: $f_n(x) = \frac{1}{nx+1}$, $x \in (0,1)$

Point wise limit:

- $x = 0 \Rightarrow f_n(0) = 1 \Rightarrow \lim_{n\to\infty} f_n(0) = 1$
- $x \in (0,1)$ $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{nx+1} = 0$

$f_n \xrightarrow{p.w.} 0$ on $(0,1)$

Uniform convergence:

take $x_n = \frac{1}{n}$, $n \geqslant 2$, $x_n \in (0,1)$

$f_n(x_n) = \frac{1}{1+1} = \frac{1}{2} \not\to 0$, $\forall n \geqslant 2$

$f_n \overset{U}{\not\to} 0$ on $(0,1)$.

on $[a,1]$, $0 < a < 1$

$\forall x \in [a,1]$

$|f_n(x)| = \frac{1}{nx+1}$

$x \in [a,1] \Rightarrow a \leqslant x \leqslant 1$

$\Rightarrow na \leqslant nx$

$\Rightarrow na+1 \leqslant nx+1$

$\Rightarrow \frac{1}{nx+1} \leqslant \frac{1}{na+1}$

$|f_n(x)| \leqslant \frac{1}{na+1}$, $\forall x \in [a,1]$, $\forall n \geqslant 2$

$\sup_{x\in[a,1]} |f_n(x)| \to 0$ as $n \to \infty$

$f_n \xrightarrow{U} 0$ on $[a,1]$

Ex 3. $f_n(x) = \frac{x}{nx+1}$, $x \in [0,1]$

Pointwise limit:

$\lim_{n\to\infty} \frac{x}{nx+1} = 0$, $\forall x \in [0,1]$

f_n Converges P.W. to 0.

Uniform convergence:

Ex: 1.4

$f_n(x) = \frac{x}{nx+1}$

Uniform convergence:

$f_n(0) = 0$

for $x \neq 0$, $f_n(x) \leqslant \frac{x}{nx} = \frac{1}{n}$

for all $x \in [0,1]$

$$|f_n(x)| \leqslant \frac{1}{n}$$

$$\sup_{x \in [0,1]} |f_n(x)| \longrightarrow 0 \text{, as } n \to \infty$$

So it's uniformly convergent.

$$f_n'(x) = \frac{nx+1-(nx)}{(nx+1)^2} = \frac{1}{(nx+1)^2} > 0$$

So f_n is increasing and the

sup is on $x = 1$

$$\Rightarrow \sup_{x \in [0,1]} f_n(x) = \frac{1}{n+1}$$

Ex 1.5:

$$f_n(x) = \frac{nx^3}{nx+1}$$

Point wise limit:

$$\lim_{n\to\infty} \frac{nx^3}{nx+1} = \begin{cases} x^2, & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

$$= x^2, \quad x \in [0,1].$$

$$f_n \xrightarrow{p.w} x^2, \text{ on } [0,1]$$

uniform convergence:

$$|f_n(x) - f(x)| = \left|\frac{nx^3}{nx+1} - x^2\right|$$
$$= \left|\frac{nx^3 - (nx^3 + x^2)}{nx+1}\right|$$
$$= \frac{x^2}{nx+1}, \quad x \in [0,1]$$

$$\begin{cases} x = 0, & |f_n(0) - f(0)| = 0 \\ x \in (0,1], & |f_n(x) - f(x)| = \frac{x^2}{nx+1} \leqslant \frac{x^2}{nx} = \frac{x}{n} \leqslant \frac{1}{n} \end{cases}$$

$$\Rightarrow \sup_{x \in [0,1]} |f_n(x) - f(x)| \leqslant \frac{1}{n}$$

$$\text{So } \lim_{n\to\infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| = 0$$

$\Rightarrow f_n$ converges uniformly to f on $[0,1]$

$[f(x) = x^2, x \in [0,1]]$

Ex 1.6:

$$f_n(x) = x^n(1-x), \quad x \in [0,1]$$

point wise limit:

$$\lim_{n\to\infty} x^n(1-x) = \begin{cases} 0, & x \in [0,1) \\ 0, & x = 1 \end{cases}$$

$$= 0, \quad x \in [0,1]$$

$$\Rightarrow \quad f_n \xrightarrow{p.w} 0, \text{ on } [0,1].$$

uniform convergence:

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup_{x \in [0,1]} |x^n(1-x)|$$
$$= \sup_{x \in [0,1]} \underbrace{x^n(1-x)}_{= g(x)}$$

$$g'(x) = \left(x^n - x^{n+1}\right)' = nx^{n-1} - (n+1)x^n$$
$$= x^{n-1}(n - (n+1)x)$$

$$g'(x) = 0 \iff x^{n-1} = 0 \text{ or } n - (n+1)x = 0$$

$$\iff x = 0 \text{ or } x = \frac{n}{n+1}$$

x	0		$\frac{n}{n+1}$		1
$g'(x)$		$+$	0	$-$	
$g(x)$		$\nearrow$		$\searrow$	