Ex 10.3

Ex 1:

let $(x_i : i \in \mathbb{N})$ be a sequence of positive real numbers

if $\mu(E)$ is defined for each $E \subseteq \mathbb{R}$

by $\mu(E) = \sum_{x_i \in E} x_i$

Prove that μ is measure on $P(\mathbb{R})$.

Solution:

Recall that let X set $\mathcal{B}$ σ-algebra on X

a function $\mu : \mathcal{B} \to [0, \infty]$ is a measure if

- $\mu(\Phi) = 0$
- if (E_i) is a countable collection of disjoint elements ~~subsets~~ of $\mathcal{B}$

we have $\boxed{\mu(\bigcup_i E_i) = \sum_i \mu(E_i)}$

$\mu(\emptyset) = \sum_{x_i \in \emptyset} x_i = 0$

let (E_i) be a collection of disjoint countable subsets of $\mathbb{R}$.

Put

$E = \bigcup_i E_i$, $A_k = \{x_i ; x_i \in E_k\} \subset E_k$

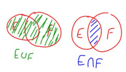
$A = \{x_i ; x_i \in E\}$

$$\mu(E) = \mu(\bigcup_i E_i) = \sum_{x_i \in E} x_i = \sum_{x_i \in A} x_i = \sum_k \sum_{x_i \in A_k} x_i = \sum_k \mu(E_k)$$

Ex 2:

for any $E, F \in \mathcal{M}$ prove that

$m(E) + m(F) = m(E \cup F) + m(E \cap F)$.



$E = (E \setminus F) \cup (E \cap F)$: disjoint union

$F = (F \setminus E) \cup (F \cap E)$: disjoint union

$E \cup F = (E \setminus F) \cup (F \setminus E) \cup (E \cap F)$: ......

Since m is a measure

then $m(E) = m(E \setminus F) + m(E \cap F)$ (1)

$m(F) = m(F \setminus E) + m(F \cap E)$ (2)

$m(E \cup F) = m(E \setminus F) + m(F \setminus E) + m(E \cap F)$ (3)

Summing (1) and (2) gives

$$m(E) + m(F) = m(E \setminus F) + m(F \setminus E) + 2m(E \cap F) = m(E \cup F) + m(E \cap F).$$

Ex 5:

if (E_n) is a sequence in $\mathcal{M}$

Prove that $m(\liminf E_n) \leq \liminf m(E_n)$

Assuming that $m(\bigcup_{i=N}^{\infty} E_i) < \infty$, for some N

Prove that $m(\limsup E_n) \geq \limsup m(E_n)$

Recall:

1- $\liminf a_n = \sup_n \inf_{k \geq n} a_k$ (increasing)

2- $\limsup a_n = \inf_n \sup_{k \geq n} a_k$ (decreasing)

3- $\liminf E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k$ (increasing)

4- $\limsup E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_n$ (decreasing)

5)

a) if $(E_i) \nearrow$ then

$$m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \to \infty} m(E_n)$$

b) if $(E_i) \searrow$ then

$$m(\bigcap_{n=1}^{\infty} E_n) = \lim_{n \to \infty} m(E_n)$$

Example

$$m\left(\bigcap_{n=1}^{\infty} [1, 1+\tfrac{1}{n}]\right) = ?$$

Since $[1, 1+\frac{1}{n+1}] \subset [1, 1+\frac{1}{n}]$

By continuity below of Lebesgue measure,

$$m\left(\bigcap_{n=1}^{\infty} [1, 1+\tfrac{1}{n}]\right) = \lim_{n \to \infty} m\left([1, 1+\tfrac{1}{n}]\right) = \lim_{n \to \infty} \frac{1}{n} = 0$$

Solution of Ex 5:

$m(\liminf E_n) = m(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k)$

Put $F_n = \bigcap_{k=n}^{\infty} E_k$, $\because F_n \subset F_{n+1}$

so $(F_n) \uparrow$

By continuity above theorem

we have $m(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_n) = m(\bigcup_{n=1}^{\infty} F_n)$

$$= \lim_{n \to \infty} m(F_n)$$

$$= \lim_{n \to \infty} m(\bigcap_{k=n}^{\infty} E_k)$$

$\bigcap_{k=n}^{\infty} E_k \subseteq E_m$, $\forall m \geq n$

Then $m\left(\bigcap_{k=n}^{\infty} E_k\right) \leq m(E_m)$, $\forall m \geq n$

$$\leq \liminf_{n \to \infty} m(E_n)$$

Exercises 11.3.

Ex 1:

evaluate $\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\, dx$

$f_n(x) = \left(1+\frac{x}{n}\right)^n e^{-2x} \chi_{[0,n]}$

- $|f_n(x)| \le e^{x} e^{-2x} = e^{-x}$, $\forall x \in [0,\infty)$, $\forall n \in \mathbb{N}$
- $\int_0^\infty e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 1 \Rightarrow e^{-x} \in L^1(0,\infty)$

- Since $f_n(x)$ is cont. on $[0,\infty)$ hence it is measurable on $[0,\infty)$
- $\lim_{n\to\infty} f_n(x) = e^{-x}$

So by dominated convergence theorem

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty \lim_{n\to\infty} f_n(x)\, dx = \int_0^\infty e^{-x}\, dx = 1$$

Therefore

$$\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\, dx = \lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = 1$$

Ex 2:

Prove that $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0$, where

a) $f_n(x) = \frac{n\sqrt{x}}{1+n^2x^2}$

b) $f_n(x) = \frac{nx\ln x}{1+n^2x^2}$

Sol. a):

- f_n is cont. on $[0,1]$. So it is measurable on $[0,1]$
- $f_n \xrightarrow{P.W} 0$ on $[0,1]$ a.e.
- $|f_n(x)| = \frac{n\sqrt{x}}{1+n^2x^2} = \frac{nx}{1+n^2x^2} \cdot \frac{1}{\sqrt{x}} \le \frac{1}{2}\frac{1}{\sqrt{x}}$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0} \left[+2\sqrt{x}\right]_\varepsilon^1$$
$$= \lim_{\varepsilon\to 0} \left(2 - 2\sqrt{\varepsilon}\right) = 2 \Rightarrow \frac{1}{2\sqrt{x}} \in L^1(0,1)$$

So by dominated convergence theorem

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0$$

Sol. b): $f_n = \frac{nx\ln x}{1+n^2x^2}$

- f_n is cont on $(0,1]$ So it is measurable a.e on $[0,1]$
- $f_n \xrightarrow{P.W} 0$ on $[0,1]$ a.e.
- $|f_n(x)| = \left|\frac{nx\ln x}{1+n^2x^2}\right| = \frac{nx|\ln x|}{1+n^2x^2} \le \frac{-\ln x}{2}$

$$\int_0^1 \ln x\, dx = \lim_{\varepsilon\to 0} \int_\varepsilon^1 \ln x\, dx$$
$$= \lim_{\varepsilon\to 0} \left[x\ln x - x\right]_\varepsilon^1$$
$$= \lim_{\varepsilon\to 0} \left[0 - 1 - (\varepsilon\ln\varepsilon - \varepsilon)\right]$$
$$= -1 - \lim_{\varepsilon\to 0} \varepsilon\ln\varepsilon$$
$$= -1 - \lim_{\varepsilon\to 0} \frac{\ln\varepsilon}{\frac{1}{\varepsilon}}$$
$$= -1 - \lim_{\varepsilon\to 0} \frac{\frac{1}{\varepsilon}}{\frac{-1}{\varepsilon^2}} = -1$$

$\frac{-\ln x}{2}$ is integrable on $[0,1]$

So by DCT

$$\lim_{n\to\infty} \int_0^1 \frac{nx\ln x}{1+n^2x^2}\,dx = \int_0^1 \lim_{n\to\infty} \frac{nx\ln x}{1+n^2x^2}\,dx = \int_0^1 0\,dx = 0$$

Ex 3: Evaluate

$\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\frac{x}{n}\,dx$; $f_n(x) = \left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right)$

$g(t) = \left(1+\frac{x}{t}\right)^{-t}$, $t>0$
$\quad = e^{-t\ln(1+\frac{x}{t})}$

$g'(t) = \left[\left(-\ln\left(1+\frac{x}{t}\right)\right) + \cancel{t}\frac{\frac{x}{t^2}}{1+\frac{x}{t}}\right]\left(1+\frac{x}{t}\right)^{-t}$

Put $h(x) = -\ln(1+x) + \frac{x}{1+x}$

$h'(x) = \frac{-1}{1+x} - \frac{1}{(1+x)^2} \le 0$

So $h(x)$ is decreasing

max $h(x)$ is at $x=0$

$h(0) = 0$

So $h(x) = -\ln|1+x| + \frac{x}{1+x} \le 0 \quad \forall x \ge 0$

$\Rightarrow g'(t) = h\left(\frac{x}{t}\right)\left(1+\frac{x}{t}\right)^{-t} \le 0$

So $g(t)$ is decreasing

So $g(n) = \left(1+\frac{x}{n}\right)^{-n} \le g(2) = \frac{1}{\left(1+\frac{x}{2}\right)^2}$

therefore

- $|f_n(x)| \le \frac{1}{\left(1+\frac{x}{2}\right)^2}$; $\forall n \ge 2$, $\forall x \ge 0$
- $\int_0^\infty \frac{1}{\left(1+\frac{x}{2}\right)^2}\,dx = \frac{1}{2}\left[\frac{-1}{1+\frac{x}{2}}\right]_0^\infty = \frac{1}{2}$
- f_n is measurable on $[0,\infty)$
- $f_n \xrightarrow{P.W} e^{-x}\cdot 0 = 0$ a.e. on $[0,\infty)$

by DCT

$$\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\frac{x}{n}\,dx = \int_0^\infty \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{-n} \sin\frac{x}{n}\,dx = \int_0^\infty 0\,dx = 0$$

Ex 4:

Prove that, for all $r>0$

$$\lim_{n\to\infty} \int_0^n \left(1-\frac{x}{n}\right)^n x^{r-1}\,dx = \int_0^\infty e^{-x} x^{r-1}\,dx$$

$\Gamma(r) = \int_0^\infty e^{-x} x^{r-1}\,dx$, $r>0$

Gamma function

Why the integral $\int_0^\infty e^{-x} x^{r-1}\,dx < \infty$

$$\int_0^\infty e^{-x} x^{r-1}\,dx = \int_0^1 e^{-x} x^{r-1}\,dx + \int_1^\infty e^{-x} x^{r-1}\,dx$$

The first integral $\int_0^1 e^{-x} x^{r-1}\,dx$ is improper near 0

$$\int_0^1 e^{-x} x^{r-1}\,dx = \int_{(0,1]} e^{-x} x^{r-1}\,dx \le \int_{(0,1]} x^{r-1}\,dx \quad (e^{-x} \le 1)$$
$$= \int_0^\infty x^{r-1} \chi_{(0,1]}(x)\,dx$$

By monotone convergence theorem

increasing in n

$$= \lim_{n\to\infty} \int_0^\infty x^{r-1} \chi_{[\frac{1}{n},1]}(x)\,dx$$

$\int_0^\infty \lim_{n\to\infty} x^{r-1}\chi_{[\frac{1}{n},1]}(x)\,dx = \lim_{n\to\infty} \int_{[\frac{1}{n},1]} x^{r-1}\,dx$ (Lebesgue integral

$= \lim_{n\to\infty} \int_{\frac{1}{n}}^1 x^{r-1}\,dx$ (Riemann. integ

$= \lim_{n\to\infty} \left[\frac{x^r}{r}\right]_{\frac{1}{n}}^1$ (Fundamental th of Calculus)

$= \lim_{n\to\infty} \left(\frac{1}{r} - \frac{1}{r}\left(\frac{1}{n}\right)^r\right) = \frac{1}{r}$, $r>0$

$\boxed{\int_0^1 x^{r-1}e^{-x}\,dx < \infty}$

The integral $\int_1^\infty e^{-x} x^{r-1}\,dx$

The function $x \to e^{-x/2}x^{r-1}$ is continuous on $[1,\infty)$

and $\lim_{x\to\infty} e^{-x/2}x^{r-1} = 0$

So $x \to e^{-x/2}x^{r-1}$ is bounded on $[1,\infty)$

then there exists $M>0$, s.t

$0 \le e^{-x/2}x^{r-1} \le M$; $\forall x \in [1,\infty)$

Therefore $0 \le e^{-x}x^{r-1} \le e^{-x/2}e^{-x/2}x^{r-1} \le M e^{-x/2}$

$\int_1^\infty e^{-x/2}\,dx = -2e^{-x/2}\Big]_0^\infty = \boxed{2}$

This shows that the function $e^{-x}x^{r-1}$ is integrable on $[1,\infty)$

$$\lim_{n\to\infty} \int_0^n \left(1-\frac{x}{n}\right)^n x^{r-1}\,dx$$

Put $f_n(x) = \left(1-\frac{x}{n}\right)^n x^{r-1} \chi_{[0,n)}$

$n \in \mathbb{N},\ x \in [0,\infty)$

- f_n is measurable on $[0,\infty)$, for all n
- (f_n) is increasing sequence. (i.e. $f_n(x) \le f_{n+1}(x)$)
- $f_n \xrightarrow{p.w} e^{-x} x^{r-1}$ on $[0,\infty)$

By monotone Convergence theorem.

$$\lim_{n\to\infty} \int_0^n \left(1-\frac{x}{n}\right)^n x^{r-1}\,dx = \lim_{n\to\infty} \int_0^\infty f_n(x)\,dx$$

$$= \int_0^\infty \lim f_n(x)\,dx$$

$$= \int_0^\infty e^{-x} x^{r-1}\,dx$$

$$= \Gamma(r)$$

$$\lim_{n\to\infty} \left(1-\frac{x}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1-\frac{x}{n}\right)}$$

Since.

$$= \lim_{n\to\infty} e^{-\frac{\ln\left(1-\frac{x}{n}\right)}{-\frac{x}{n}}\cdot x} = e^{-x}$$

$$\lim_{t\to 0} \frac{\ln(1-t)}{-t} = \lim_{t\to 0} \frac{\frac{-1}{1-t}}{-1} = \lim_{t\to 0} \frac{1}{1-t} = 1$$