# Math 106
# Integral Calculus
## Exercises Chapter 9 (Parametric Equations and Polar Coordinates)

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# Exercises

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2. 9.2 Arc Length and Surface Area
3. 9.3 Polar Coordinates
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   - Area
   - Arc Length

# 1.1 Parametric Equations

# Parametric Equations

**Definition**

A plane curve is a set of ordered pairs $(f(t), g(t))$, where $f$ and $g$ are continuous on an interval $I$.

**Definition**

Let $C$ be a curve consists of all ordered pairs $(f(t), g(t))$ , where $f$ and $g$ are continuous on an interval $I$. The equations

$$x = f(t), \qquad y = g(t)$$

for $t \in I$ are parametric equations for $C$ with parameter $t$.

# Parametric Equations

## Example

Sketch the graph of $C$ where $C$ is the curve

$$x = t - 2, \qquad y = 2t + 3, \qquad 0 \leq t \leq 5$$

**Example**

Sketch the graph of $C$ where $C$ is the curve

$$x = t^2 + 1, \qquad y = t^2 - 1, \qquad -1 \leq t \leq 4$$

## Example

Sketch the graph of $C$ where $C$ is the curve

$$x = 4t^2 - 5, \qquad y = 2t + 3, \qquad t \in \mathbb{R}$$

# Parametric Equations

## Example

Sketch the graph of $C$ where $C$ is the curve

$$x = e^t, \qquad y = e^{-2t}, \qquad t \in \mathbb{R}$$

# 9.2 Arc Length and Surface Area

# Slope of a Curve

**Theorem**

*If a smooth curve $C$ is given parametrically by $x = f(t), y = g(t)$ then the slope $\frac{dy}{dx}$ of the tangent line to $C$ at a point $P(x, y)$ is*

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \frac{dx}{dt} \neq 0$$

# Arc Length

## Theorem

*If a smooth curve* $C$ *is given parametrically by* $x = f(t), y = g(t)$*;* $a \leq t \leq b$ *and* $C$ *does not intersect itself, except possibly for* $t = a$ *and* $t = b$*. Then the length* $L$ *of* $C$ *is*

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

# Surface Area

**Theorem**

*If a smooth curve $C$ is given parametrically by $x = f(t), y = g(t)$; $a \leq t \leq b$ and $C$ does not intersect itself, except possibly for $t = a$ and $t = b$. If $g(t) \geq 0$, then the area $S$ of the surface of revolution obtained by revolving $C$ about the $x-$axis is*

$$S = \int_a^b 2\pi g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

**Theorem**

*If $f(t) \geq 0$, then the area $S$ of the surface of revolution obtained by revolving $C$ about the $y-$axis is*

$$S = \int_a^b 2\pi f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

# Slope of a Curve

## Example

Find the slope of the tangent line of the curve

$$x = t^2 + 1, \qquad y = t^2 - 1 \qquad -2 \leq t \leq 2$$

at $t = 1$

# Slope of a Curve

## Example

Find the point on curve

$$x = -t^3, \qquad y = -6t^2 - 18t, \qquad t \geq 0$$

at which the slope is $2$

# Arc Length

## Example

Find the arc length of the curve

$$x = 5t^2, \quad y = 2t^3, \quad 0 \leq t \leq 1$$

# Surface Area

## Example

Find the surface area of the solid obtained by revolving the curve

$$x = t^2,\ y = 2t,\ 0 \leq t \leq 4$$

about the $x$-axis

# Surface Area

## Example

Find the surface area of the solid obtained by revolving the curve

$$x = t - \sin t,\ y = 1 - \cos t,\ 0 \leq t \leq 2\pi$$

about the $x$-axis

# Surface Area

## Example

Find the surface area obtained by revolving the curve $x = 4\sqrt{t}$, $y = \frac{1}{2}t^2 + t^{-1}, 1 \leq t \leq 4$ about the $y$-axis

# Arc Length

## Exam problem

Find the arc length of the parametric curve $x = \dfrac{t^4}{4}\ ,\quad y = \dfrac{t^6}{6}\ ,\quad 0 \leq t \leq 1$

# 9.3 Polar Coordinates

# Polar and Rectangular Coordinates

The rectangular coordinates $(x, y)$ and polar coordinates $(r, \theta)$ of a point $P$ are related as follows:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}, \qquad x \neq 0$$

# Functions in Polar Coordinates

## Example

Plot the functions

1. $r = 5$
2. $r = -2$
3. $\theta = -\frac{\pi}{6}$
4. $r = 3\cos\theta$

# Circle

## Example

Plot the function

$$r = 4 - 4\sin\theta$$

# Cardioid

## Example

Find a polar equation

1. $x = 3$
2. $x^2 + y^2 = 16$
3. $2y = -x$
4. $y^2 - x^2 = 4$

# Cardioid

## Example

Find an equation in $x$ and $y$

1. $r\cos\theta = 5$
2. $r\sin\theta = -2$

# 9.4 Integrals In Polar Coordinates

# Area

## Theorem

*If $f$ is continuous and $f(\theta) \geq 0$ on $[a, b]$, where $0 \leq a < b \leq 2\pi$, then the area $A$ of the region bounded by the graph of $r = f(\theta)$, $\theta = a$, and $\theta = b$ is*

$$A = \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta = \int_a^b \frac{1}{2} r^2 \, d\theta$$

# Arc Length

To find the arc length of $r = f(\theta)$ from $\theta = a$ to $\theta = b$, we use parametric equations

$$x = f(\theta)\cos\theta, \qquad y = f(\theta)\sin\theta, \qquad a \leq \theta \leq b$$

## Theorem

*The arc length of $f(\theta)$ from $\theta = a$, to $\theta = b$ is*

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

# Surface of Revolution

## Theorem

*The surface of revolution of $f(\theta)$ from $\theta = a$, to $\theta = b$ about the polar axis is*

$$S = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

*and about the line $\theta = \frac{\pi}{2}$ is*

$$S = \int_a^b 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

# Area

## Example

Find the area of the region bounded by

$$r = 2\cos\theta$$

**Example**

Find the area of the region bounded by $r = 1 - \cos\theta$.

# Area

## Example

Sketch the region inside $r = 2 + 2\cos\theta$ and inside $r = 6\cos\theta$ and find its area

## Example

Sketch the region inside $r = 2 + 2\cos\theta$ and outside $r = 6\cos\theta$ and find its area

# Area

## Example

Sketch the region outside $r = 1 - \sin\theta$ and inside $r = 3\sin\theta$ and find its area

# Arc Length

## Example

Find the length of the curve $r = e^{-\theta}$ from $\theta = 0$ to $\theta = 2\pi$

# Surface of Revolution

## Example

Find the area of the surface of revolution of $r = 2 + 2\cos\theta$ from $\theta = 0$, to $\theta = \frac{\pi}{2}$ about the polar axis.

# Area

**Exam problem**

Sketch the region inside $r = 3 + 3\sin\theta$ and outside $r = 3$ and find its area