

Math 106

Integral Calculus

Exercises Chapter 7 (Techniques of Integration)

Ibraheem Alolyan

King Saud University

- 1 7.1 Integration By Parts
- 2 7.2 Trigonometric Integrals
- 3 7.3 Trigonometric Substitutions
- 4 7.4 Integrals of Rational Functions
(Partial Fraction Decomposition)
- 5 7.5 Quadratic Forms and Miscellaneous Substitution
- 6 7.7 Improper Integrals

7.1 Integration By Parts

Integration By Parts

Theorem

If $u = f(x)$ and $v = g(x)$ and if f' and g' are continuous then

$$\int u dv = uv - \int v du$$

Integration By Parts

Example

Evaluate the following integrals

① $\int x e^{-x} dx$

② $\int x \sin x dx$

③ $\int x \sec x \tan x dx$

Integration By Parts

Example

Evaluate the following integrals

① $\int \tan^{-1} x \, dx$

② $\int \sin^{-1} x \, dx$

③ $\int \sqrt{x} \ln x \, dx$

Integration By Parts

Example

Evaluate the following integrals

① $\int x \csc^2 x \, dx$

② $\int e^{-x} \sin x \, dx$

③ $\int (\ln x)^2 \, dx$

Exam Problem

Exam problem

Find the indefinite integral $\int x(\cos x)^2 dx$

Exam Problem

Exam problem

Find the indefinite integral $\int x(\ln x)^2 dx$

7.2 Trigonometric Integrals

Trigonometric Integrals

To evaluate

$$\int \sin^m x \cos^n x dx$$

- 1 If m is an odd integer

$$\int \sin^m x \cos^n x dx = \int \sin^{m-1} x \cos^n x \sin x dx$$

Then use the identity $\sin^2 x = 1 - \cos^2 x$, and the substitution $u = \cos x$

Trigonometric Integrals

- 2 If n is an odd integer

$$\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{n-1} x \cos x dx$$

Then use the identity $\cos^2 x = 1 - \sin^2 x$, and the substitution $u = \sin x$

- 3 If m and n is are even integer Then use the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Trigonometric Integrals

To evaluate

$$\int \tan^m x \sec^n x dx$$

- 1 If m is an odd integer

$$\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x dx$$

Then use the identity $\tan^2 x = \sec^2 x - 1$, and the substitution $u = \sec x$

Trigonometric Integrals

- ② If n is an even integer

$$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x \sec^2 x dx$$

Then use the identity $\sec^2 x = 1 + \tan^2 x$, and the substitution $u = \sec x$

- ③ If m is even and n is odd integers, we have to find another way, sometimes integration by parts.

Trigonometric Integrals

To evaluate

$$\int \sin mx \cos nx \, dx, \quad \int \sin mx \sin nx \, dx$$

$$\int \cos mx \cos nx \, dx$$

we use the following formulas

$$\sin nx \cos mx = \frac{1}{2}[\sin(n - m)x + \sin(n + m)x]$$

$$\cos nx \cos mx = \frac{1}{2}[\cos(n - m)x + \cos(n + m)x]$$

$$\sin nx \sin mx = \frac{1}{2}[\cos(n - m)x - \cos(n + m)x]$$

Trigonometric Integrals

Example

Evaluate the following integrals

① $\int \cos^3 x \, dx$

② $\int \sin^2 x \cos^2 x \, dx$

③ $\int \cos^7 x \, dx$

Trigonometric Integrals

Example

Evaluate the following integrals

① $\int \sin^3 x \cos^2 x \, dx$

② $\int \sin^6 x \, dx$

Trigonometric Integrals

Example

Evaluate the following integrals

① $\int \tan^3 x \sec^4 x dx$

② $\int \tan^3 x \sec^3 x dx$

③ $\int \tan^6 x dx$

Trigonometric Integrals

Example

Evaluate the following integrals

① $\int \sqrt{\sin x} \cos^3 x \, dx$

Exam Problem

Exam problem

Compute $\int \sin(4x) \cos(2x) dx$

Exam Problem

Exam problem

Evaluate $\int \sin^4(2x) dx$

7.3 Trigonometric Substitutions

Trigonometric Substitutions

For $\sqrt{a^2 - x^2}$, use

$$x = a \sin \theta, \quad \theta \in [-\pi/2, \pi/2]$$

$$dx = a \cos \theta d\theta$$

$$\sqrt{a^2 - x^2} = a \cos \theta$$

For $\sqrt{a^2 + x^2}$, use

$$x = a \tan \theta, \quad \theta \in (-\pi/2, \pi/2)$$

$$dx = a \sec^2 \theta d\theta$$

$$\sqrt{a^2 + x^2} = a \sec \theta$$

For $\sqrt{x^2 - a^2}$, use

$$x = a \sec \theta, \quad \theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$$

$$dx = a \sec \theta \tan \theta d\theta$$

$$\sqrt{x^2 - a^2} = a \tan \theta$$

Trigonometric Substitutions

Example

Evaluate

$$\int \frac{1}{x\sqrt{4-x^2}} dx$$

Trigonometric Substitutions

Example

Evaluate

$$\int \frac{1}{x\sqrt{9+x^2}} dx$$

Trigonometric Substitutions

Example

Evaluate

$$\int \frac{1}{x^2 \sqrt{x^2 - 25}} dx$$

Trigonometric Substitutions

Example

Evaluate

$$\int \frac{x}{\sqrt{4-x^2}} dx$$

Trigonometric Substitutions

Example

Evaluate

$$\int \frac{1}{(x^2 - 1)^{\frac{3}{2}}} dx$$

Trigonometric Substitutions

Example

Evaluate

$$\int \frac{1}{\sqrt{4x^2 - 25}} dx$$

Trigonometric Substitutions

Example

Evaluate

$$\int \frac{(4 + x^2)^2}{x^3} dx$$

Trigonometric Substitutions

Example

Evaluate

$$\int \frac{3x - 5}{\sqrt{1 - x^2}} dx$$

Exam Problem

Exam problem

Find $\int \frac{dx}{x^3 \sqrt{x^2 - 4}}$

7.4 Integrals of Rational Functions (Partial Fraction Decomposition)

Partial Fraction Decomposition

To find the integral

$$\int q(x)dx = \int \frac{f(x)}{g(x)}dx$$

where $f(x)$ and $g(x)$ are polynomials.

- 1 If $\deg f(x) \geq \deg g(x)$, use long division.

$$q(x) = h(x) + \frac{r(x)}{g(x)}$$

where the degree of $r(x)$ is less than $g(x)$.

- 2 Use **partial fraction decomposition**,

Partial Fraction Decomposition

- 1 If $q(x) = \frac{f(x)}{g(x)}$ and degree of $f(x)$ is less than degree of $g(x)$, we factor $g(x)$ into linear polynomials

$$(ax + b)^n, \quad n \in \mathbb{N}$$

or irreducible quadratic polynomials

$$(ax^2 + bx + c)^n$$

(irreducible means the polynomial has no real zeros, i.e. $b^2 - 4ac < 0$)

Partial Fraction Decomposition

- ③ We write

$$q(x) = \frac{f(x)}{g(x)} = F_1(x) + F_2(x) + \dots + F_r(x)$$

where each $F_k(x)$ has one of the forms

$$\frac{A}{(ax + b)^n} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + c)^n}$$

The sum $F_1(x) + F_2(x) + \dots + F_r(x)$ is the **partial fraction decomposition** of $q(x)$.

Partial Fraction Decomposition

Case 1: Distinct Linear Factors If

$$g(x) = (x - a_1)(x - a_2) \cdots (x - a_n),$$

then

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n}.$$

Partial Fraction Decomposition

Case 2: Repeated Linear Factors

If $(x - a)^m$ is a factor of $g(x)$, then include terms of the form:

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \dots + \frac{A_m}{(x - a)^m}.$$

Partial Fraction Decomposition

Case 3: Irreducible Quadratic Factors If $(ax^2 + bx + c)$ is an irreducible quadratic factor, include:

$$\frac{Bx + C}{ax^2 + bx + c}.$$

If it is repeated m times:

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \dots + \frac{B_mx + C_m}{(ax^2 + bx + c)^m}.$$

Partial Fraction Decomposition

Example

Evaluate the following integrals

$$\textcircled{1} \int \frac{5x - 12}{x(x - 4)} dx$$

$$\textcircled{2} \int \frac{x + 34}{(x - 6)(x + 2)} dx$$

Partial Fraction Decomposition

Example

Evaluate the following integrals

$$\textcircled{1} \int \frac{6x - 11}{(x - 1)^2} dx$$

$$\textcircled{2} \int \frac{-19x^2 + 50x - 25}{x^2(3x - 5)} dx$$

Partial Fraction Decomposition

Example

Evaluate the following integrals

$$① \int \frac{5x^2 - 10x - 8}{x^3 - 4x} dx$$

$$② \int \frac{2x^2 - 25x - 33}{(x + 1)^2(x - 5)} dx$$

Partial Fraction Decomposition

Example

Evaluate the following integrals

$$① \int \frac{x^6 - x^3 + 1}{x^4 + 9x^2} dx$$

Exam Problem

Exam problem

Evaluate $\int \frac{x^2 + 1}{x^3 - x} dx$

7.5 Quadratic Forms and Miscellaneous Substitution

Quadratic Forms

If the integral contains irreducible quadratic expression

$$ax^2 + bx + c$$

where $b \neq 0$.

$$ax^2 + bx + c = a \left(x^2 + \frac{b}{a}x \right) + c = a \left(x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a}$$

Integrals Involving Fractional Powers

If we have integrand that consists of fractional powers, we use the substitution

$$u = x^{\frac{1}{n}}$$

where n is the least common multiple of the denominators of the powers.

$$\int \frac{1}{x^{\frac{1}{k}} + x^{\frac{1}{m}}} dx$$

substitute

$$u = x^{\frac{1}{n}}$$

where n is the least common multiple of m and k .

$$u^n = x$$

$$nu^{n-1} du = dx$$

$$x^{\frac{1}{k}} = u^{\frac{n}{k}} \qquad x^{\frac{1}{m}} = u^{\frac{n}{m}}$$

Integrals of the Form $\sqrt[n]{f(x)}$

If integral contains an expression of the form $\sqrt[n]{f(x)}$ we use the substitution

$$u = \sqrt[n]{f(x)}$$

$$u^n = f(x)$$

$$nu^{n-1}du = f'(x)dx$$

Fractional Functions in $\sin x$ and $\cos x$

If the integral consists of rational expression in $\sin x$ and $\cos x$ then we use **tangent half-angle substitution**

$$u = \tan \frac{x}{2}, \quad -\pi < x < \pi$$

then we have

$$\begin{aligned}\sin x &= \frac{2u}{u^2 + 1} \\ \cos x &= \frac{1 - u^2}{u^2 + 1} \\ dx &= \frac{2}{u^2 + 1} du\end{aligned}$$

Quadratic Forms and Miscellaneous Substitution

Example

Evaluate the following integrals

$$\textcircled{1} \int \frac{1}{(x+1)^2 + 4} dx$$

$$\textcircled{2} \int \frac{1}{x^2 - 4x + 8} dx$$

Quadratic Forms and Miscellaneous Substitution

Example

Evaluate the following integrals

$$\textcircled{1} \int \frac{1}{\sqrt{4x - x^2}} dx$$

$$\textcircled{2} \int \frac{1}{\sqrt{7 + 6x - x^2}} dx$$

Quadratic Forms and Miscellaneous Substitution

Example

Evaluate the following integrals

$$\textcircled{1} \int \frac{1}{(x^2 - 6x + 34)^{\frac{3}{2}}} dx$$

$$\textcircled{2} \int \sqrt{x(6-x)} dx$$

Quadratic Forms and Miscellaneous Substitution

Example

Evaluate the following integrals

$$\textcircled{1} \int_4^9 \frac{1}{\sqrt{x} + 4} dx$$

$$\textcircled{2} \int_0^{25} \frac{1}{\sqrt{4 + \sqrt{x}}} dx$$

Quadratic Forms and Miscellaneous Substitution

Example

Evaluate the following integrals

1 $\int \frac{\sqrt{x}}{1 + \sqrt[3]{x}} dx$

2 $\int \frac{1}{\sqrt[4]{x} + \sqrt[3]{x}} dx$

Quadratic Forms and Miscellaneous Substitution

Example

Evaluate the following integrals

$$\textcircled{1} \int \frac{x^{\frac{1}{3}} + 1}{x^{\frac{1}{3}} - 1} dx$$

$$\textcircled{2} \int \frac{1}{2 + \sin x} dx$$

Quadratic Forms and Miscellaneous Substitution

Example

Evaluate the following integrals

$$\textcircled{1} \int \frac{1}{3 + 2 \cos x} dx$$

$$\textcircled{2} \int \frac{1}{1 + \sin x + \cos x} dx$$

Quadratic Forms and Miscellaneous Substitution

Example

Evaluate the following integrals

$$\textcircled{1} \int \frac{1}{\tan x + \sin x} dx$$

Exam Problem

Exam problem

Compute $\int \frac{1}{\sqrt{x^2 - 4x + 2}} dx$

7.7 Improper Integrals

Integrals with Infinite limits of integration

Definition

- ① If f is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)dx = \lim_{t \rightarrow \infty} \int_a^t f(x)dx$$

provided the limit exists.

- ② If f is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^a f(x)dx = \lim_{t \rightarrow -\infty} \int_t^a f(x)dx$$

provided the limit exists.

Integrals with Infinite limits of integration

- An improper integral is said to **converge** if the limit exists.
- If the limit does not exist, then the improper integral is **diverges**

Integrals with Infinite limits of integration

Definition

If f is continuous on \mathbb{R} , then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^{\infty} f(x)dx$$

provided both improper integrals converge.

Integrals with discontinuous integrands

Definition

- ① If f is continuous on $[a, b)$, and discontinuous at b then

$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx$$

provided the limit exists.

- ② If f is continuous on $(a, b]$, and discontinuous at a , then

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^+} \int_t^b f(x)dx$$

provided the limit exists.

Integrals with discontinuous integrands

Definition

If f is discontinuous at $c \in (a, b)$, but continuous elsewhere, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

provided both integrals converge.

Improper Integrals

Example

Determine whether the improper integral converges or diverges

$$\int_1^{\infty} \frac{1}{x^{\frac{4}{3}}} dx$$

Improper Integrals

Example

Determine whether the improper integral converges or diverges

$$\int_{-\infty}^0 \frac{1}{(x-1)^3} dx$$

Improper Integrals

Example

Determine whether the improper integral converges or diverges

$$\int_0^{\infty} \frac{x}{1+x^2} dx$$

Improper Integrals

Example

Determine whether the improper integral converges or diverges

$$\int_0^{\infty} e^{-2x} dx$$

Improper Integrals

Example

Determine whether the improper integral converges or diverges

$$\int_0^{\infty} \frac{\cos x}{1 + \sin^2 x} dx$$

Improper Integrals

Example

Determine whether the improper integral converges or diverges

$$\int_{-\infty}^2 \frac{1}{x^2 + 1} dx$$

Improper Integrals

Example

Determine whether the improper integral converges or diverges

$$\int_{-\infty}^{\infty} x e^{-x^2} dx$$

Improper Integrals

Example

Determine whether the improper integral converges or diverges

$$\int_1^{\infty} \frac{\ln x}{x} dx$$

Improper Integrals

Exam problem

Does the integral $\int_0^1 \frac{dx}{\sqrt{x}(1+x)}$ converge? Find its value if it does