# Math 106
# Integral Calculus
## Exercises Chapter 7 (Techniques of Integration)

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# Exercises

# 7.1 Integration By Parts

# Integration By Parts

**Theorem**

*If* $u = f(x)$ *and* $v = g(x)$ *and if* $f'$ *and* $g'$ *are continuous then*

$$\int u dv = uv - \int v du$$

**Example**

Evaluate the following integrals

1. $\displaystyle\int xe^{-x}\,dx$
2. $\displaystyle\int x\sin x\,dx$
3. $\displaystyle\int x\sec x\tan x\,dx$

## Example

Evaluate the following integrals

1. $\displaystyle\int \tan^{-1} x \, dx$
2. $\displaystyle\int \sin^{-1} x \, dx$
3. $\displaystyle\int \sqrt{x} \ln x \, dx$

## Example

Evaluate the following integrals

1. $\displaystyle\int x \csc^2 x \, dx$
2. $\displaystyle\int e^{-x} \sin x \, dx$
3. $\displaystyle\int (\ln x)^2 \, dx$

# Exam Problem

## Exam problem

Find the indefinite integral $\int x(\cos x)^2 dx$

# Exam Problem

## Exam problem

Find the indefinite integral $\int x(\ln x)^2 dx$

# 7.2 Trigonometric Integrals

To evaluate

$$\int \sin^m x \, \cos^n x \, dx$$

1. If m is an odd integer

$$\int \sin^m x \, \cos^n x \, dx = \int \sin^{m-1} x \, \cos^n x \, \sin x \, dx$$

Then use the identity $\sin^2 x = 1 - \cos^2 x$, and the substitution $u = \cos x$

# Trigonometric Integrals

2. If n is an odd integer

$$\int \sin^m x \, \cos^n x \, dx = \int \sin^m x \, \cos^{n-1} x \, \cos x \, dx$$

Then use the identity $\cos^2 x = 1 - \sin^2 x$, and the substitution $u = \sin x$

3. If m and n is are even integer Then use the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

# Trigonometric Integrals

To evaluate

$$\int \tan^m x \, \sec^n x \, dx$$

1. If m is an odd integer

$$\int \tan^m x \, \sec^n x \, dx = \int \tan^{m-1} x \, \sec^{n-1} x \, \sec x \tan x \, dx$$

Then use the identity $\tan^2 x = \sec^2 x - 1$, and the substitution $u = \sec x$

2. If n is an even integer

$$\int \tan^m x \, \sec^n x \, dx = \int \tan^m x \, \sec^{n-2} x \, \sec^2 x \, dx$$

Then use the identity $\sec^2 x = 1 + \tan^2 x$, and the substitution $u = \sec x$

3. If m is even and n is odd integers, we have to find another way, sometimes integration by parts.

To evaluate

$$\int \sin mx \, \cos nx \, dx, \qquad \int \sin mx \, \sin nx \, dx$$

$$\int \cos mx \, \cos nx \, dx$$

we use the following formulas

$$\sin nx \, \cos mx = \frac{1}{2}[\sin(n-m)x + \sin(n+m)x]$$

$$\cos nx \, \cos mx = \frac{1}{2}[\cos(n-m)x + \cos(n+m)x]$$

$$\sin nx \, \sin mx = \frac{1}{2}[\cos(n-m)x - \cos(n+m)x]$$

# Trigonometric Integrals

## Example

Evaluate the following integrals

1. $\displaystyle\int \cos^3 x\,dx$
2. $\displaystyle\int \sin^2 x\cos^2 x\,dx$
3. $\displaystyle\int \cos^7 x\,dx$

# Trigonometric Integrals

## Example

Evaluate the following integrals

1. $\int \sin^3 x \cos^2 x \, dx$
2. $\int \sin^6 x \, dx$

# Trigonometric Integrals

## Example

Evaluate the following integrals

1. $$\int \tan^3 x \sec^4 x \, dx$$
2. $$\int \tan^3 x \sec^3 x \, dx$$
3. $$\int \tan^6 x \, dx$$

## Example

Evaluate the following integrals

1. $\int \sqrt{\sin x} \cos^3 x \, dx$

# Exam Problem

## Exam problem

Compute $\int \sin(4x)\cos(2x)dx$

# Exam Problem

## Exam problem

Evaluate $\int \sin^4(2x)dx$

# 7.3 Trigonometric Substitutions

# Trigonometric Substitutions

For $\sqrt{a^2 - x^2}$, use

$$x = a\sin\theta, \qquad \theta \in [-\pi/2, \pi/2]$$

$$dx = a\cos\theta\, d\theta$$

$$\sqrt{a^2 - x^2} = a\cos\theta$$

For $\sqrt{a^2 + x^2}$, use

$$x = a\tan\theta, \qquad \theta \in (-\pi/2, \pi/2)$$

$$dx = a\sec^2\theta\, d\theta$$

$$\sqrt{a^2 + x^2} = a\sec\theta$$

For $\sqrt{x^2 - a^2}$, use

$$x = a\sec\theta, \qquad \theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$$

$$dx = a\sec\theta\tan\theta\, d\theta$$

$$\sqrt{x^2 - a^2} = a\tan\theta$$

# Trigonometric Substitutions

## Example

Evaluate

$$\int \frac{1}{x\sqrt{4-x^2}}\, dx$$

# Trigonometric Substitutions

## Example

Evaluate

$$\int \frac{1}{x\sqrt{9+x^2}}\,dx$$

# Trigonometric Substitutions

## Example

Evaluate

$$\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$$

# Trigonometric Substitutions

## Example

Evaluate

$$\int \frac{x}{\sqrt{4-x^2}}\,dx$$

## Example

Evaluate

$$\int \frac{1}{(x^2-1)^{\frac{3}{2}}}\,dx$$

# Trigonometric Substitutions

## Example

Evaluate

$$\int \frac{1}{\sqrt{4x^2 - 25}}\, dx$$

## Example

Evaluate

$$\int \frac{(4+x^2)^2}{x^3}\,dx$$

## Example

Evaluate

$$\int \frac{3x-5}{\sqrt{1-x^2}}\,dx$$

# Exam Problem

## Exam problem

Find $\displaystyle\int \frac{dx}{x^3\sqrt{x^2-4}}$

# 7.4 Integrals of Rational Functions (Partial Fraction Decomposition)

# Partial Fraction Decomposition

To find the integral

$$\int q(x)dx = \int \frac{f(x)}{g(x)}dx$$

where $f(x)$ and $g(x)$ are polynomials.

1. If $\deg f(x) \geq \deg g(x)$, use long division.

$$q(x) = h(x) + \frac{r(x)}{g(x)}$$

   where the degree of $r(x)$ is less than $g(x)$.

2. Use partial fraction decomposition,

# Partial Fraction Decomposition

1. If $q(x) = \frac{f(x)}{g(x)}$ and degree of $f(x)$ is less than degree of $g(x)$, we factor $g(x)$ into linear polynomials

$$(ax+b)^n, \qquad n \in \mathbb{N}$$

or irreducible quadratic polynomials

$$(ax^2+bx+c)^n$$

( irreducible means the polynomial has no real zeros, i.e. $b^2 - 4ac < 0$)

3. We write

$$q(x) = \frac{f(x)}{g(x)} = F_1(x) + F_2(x) + \ldots + F_r(x)$$

where each $F_k(x)$ has one of the forms

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^n}$$

The sum $F_1(x) + F_2(x) + \ldots + F_r(x)$ is the **partial fraction decomposition** of $q(x)$.

# Partial Fraction Decomposition

**Case 1: Distinct Linear Factors** If

$$g(x) = (x - a_1)(x - a_2)\cdots(x - a_n),$$

then

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n}.$$

# Partial Fraction Decomposition

**Case 2: Repeated Linear Factors**
If $(x-a)^m$ is a factor of $g(x)$, then include terms of the form:

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m}.$$

# Partial Fraction Decomposition

**Case 3: Irreducible Quadratic Factors** If $(ax^2 + bx + c)$ is an irreducible quadratic factor, include:

$$\frac{Bx + C}{ax^2 + bx + c}.$$

If it is repeated $m$ times:

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_mx + C_m}{(ax^2 + bx + c)^m}.$$

# Partial Fraction Decomposition

## Example

Evaluate the following integrals

1. $$\int \frac{5x-12}{x(x-4)}\,dx$$
2. $$\int \frac{x+34}{(x-6)(x+2)}\,dx$$

# Partial Fraction Decomposition

## Example

Evaluate the following integrals

1. $$\int \frac{6x-11}{(x-1)^2}\,dx$$
2. $$\int \frac{-19x^2+50x-25}{x^2(3x-5)}\,dx$$

## Example

Evaluate the following integrals

1. $$\int \frac{5x^2 - 10x - 8}{x^3 - 4x}\,dx$$
2. $$\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)}\,dx$$

# Partial Fraction Decomposition

## Example

Evaluate the following integrals

1. $\displaystyle\int \frac{x^6 - x^3 + 1}{x^4 + 9x^2}\,dx$

# Exam Problem

## Exam problem

Evaluate $\int \frac{x^2+1}{x^3-x}dx$

# Quadratic Forms and Miscellaneous Substitution

# Quadratic Forms

If the integral contains irreducible quadratic expression

$$ax^2 + bx + c$$

where $b \neq 0$.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

# Integrals Involving Fractional Powers

If we have integrand that consists of fractional powers, we use the substitution

$$u = x^{\frac{1}{n}}$$

where $n$ is the least common multiple of the denominators of the powers.

$$\boxed{\int \frac{1}{x^{\frac{1}{k}} + x^{\frac{1}{m}}}\, dx}$$

substitute

$$\boxed{u = x^{\frac{1}{n}}}$$

where $n$ is the least common multiple of $m$ and $k$.

$$u^n = x$$

$$nu^{n-1}du = dx$$

$$x^{\frac{1}{k}} = u^{\frac{n}{k}} \qquad x^{\frac{1}{m}} = u^{\frac{n}{m}}$$

# Integrals of the Form $\sqrt[n]{f(x)}$

If integral contains an expression of the form $\sqrt[n]{f(x)}$ we use the substitution

$$u = \sqrt[n]{f(x)}$$
$$u^n = f(x)$$
$$nu^{n-1}du = f'(x)dx$$

# Fractional Functions in $\sin x$ and $\cos x$

If the integral consists of rational expression in $\sin x$ and $\cos x$ then we use tangent half-angle substitution

$$u = \tan \frac{x}{2}, \qquad -\pi < x < \pi$$

then we have

$$\sin x = \frac{2u}{u^2+1}$$
$$\cos x = \frac{1-u^2}{u^2+1}$$
$$dx = \frac{2}{u^2+1} du$$

# Quadratic Forms and Miscellaneous Substitution

## Example

Evaluate the following integrals

1. $$\int \frac{1}{(x+1)^2+4}\,dx$$
2. $$\int \frac{1}{x^2-4x+8}\,dx$$

# Quadratic Forms and Miscellaneous Substitution

## Example

Evaluate the following integrals

1. $\displaystyle\int \frac{1}{\sqrt{4x - x^2}}\, dx$
2. $\displaystyle\int \frac{1}{\sqrt{7 + 6x - x^2}}\, dx$

## Example

Evaluate the following integrals

1. $$\int \frac{1}{(x^2 - 6x + 34)^{\frac{3}{2}}}\, dx$$
2. $$\int \sqrt{x(6 - x)}\, dx$$

## Example

Evaluate the following integrals

1. $$\int_4^9 \frac{1}{\sqrt{x}+4}\,dx$$
2. $$\int_0^{25} \frac{1}{\sqrt{4+\sqrt{x}}}\,dx$$

## Example

Evaluate the following integrals

1. $$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$$
2. $$\int \frac{1}{\sqrt[4]{x}+\sqrt[3]{x}}\,dx$$

## Example

Evaluate the following integrals

1. $$\int \frac{x^{\frac{1}{3}}+1}{x^{\frac{1}{3}}-1}\,dx$$
2. $$\int \frac{1}{2+\sin x}\,dx$$

## Example

Evaluate the following integrals

1. $$\int \frac{1}{3 + 2\cos x}\,dx$$
2. $$\int \frac{1}{1 + \sin x + \cos x}\,dx$$

## Example

Evaluate the following integrals

1. $\displaystyle\int \frac{1}{\tan x + \sin x}\,dx$

# Exam Problem

## Exam problem

Compute $\int \frac{1}{\sqrt{x^2 - 4x + 2}} dx$

# Improper Integrals

# Integrals with Infinite limits of integration

## Definition

1. If $f$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided the limit exists.

2. If $f$ is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^a f(x)dx = \lim_{t\to-\infty} \int_t^a f(x)dx$$

provided the limit exists.

# Integrals with Infinite limits of integration

- An improper integral is said to **converge** if the limit exists.
- If the limit does not exist, then the improper integral is **diverges**

# Integrals with Infinite limits of integration

## Definition

If $f$ is continuous on $\mathbb{R}$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

provided both improper integrals converge.

# Integrals with discontinuous integrands

## Definition

1. If $f$ is continuous on $[a, b)$, and discontinuous at $b$ then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

provided the limit exists.

2. If $f$ is continuous on $(a, b]$, and discontinuous at $a$ , then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^a f(x)dx$$

provided the limit exists.

# Integrals with discontinuous integrands

## Definition

If $f$ is discontinuous at $c \in (a, b)$, but continuous elsewhere, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

provided both integrals converge.

## Example

Determine whether the improper integral converges of diverges

$$\int_1^{\infty} \frac{1}{x^{\frac{4}{3}}} dx$$

**Example**

Determine whether the improper integral converges of diverges

$$\int_{-\infty}^{0} \frac{1}{(x-1)^3}\,dx$$

## Example

Determine whether the improper integral converges of diverges

$$\int_0^\infty \frac{x}{1+x^2}dx$$

## Example

Determine whether the improper integral converges of diverges

$$\int_0^{\infty} e^{-2x} dx$$

# Improper Integrals

## Example

Determine whether the improper integral converges of diverges

$$\int_0^{\infty} \frac{\cos x}{1+\sin^2 x} dx$$

## Example

Determine whether the improper integral converges of diverges

$$\int_{-\infty}^{2} \frac{1}{x^2+1} dx$$

# Improper Integrals

## Example

Determine whether the improper integral converges of diverges

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$$

## Example

Determine whether the improper integral converges of diverges

$$\int_1^{\infty} \frac{\ln x}{x} dx$$

## Exam problem

Does the integral $\int_0^1 \frac{dx}{\sqrt{x}(1+x)}$ converge ? Find its value if it does