Math 106
Integral Calculus
Exercises Chapter 6 (Logarithm and Exponential Functions)

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Exercises

6.2 Natural Logarithm Function

Natural Logarithm Function

Example

Find $f'(x)$ where

1. $f(x) = \ln(3x^2 - 2x + 1)$
2. $f(x) = \ln|4 - x|$
3. $f(x) = x \ln x$
4. $f(x) = \dfrac{1}{\ln x} + \ln \dfrac{1}{x}$

Natural Logarithm Function

Example

Find $f'(x)$ where

1. $f(x) = \ln |\sqrt[3]{4x-5}\,(3x+8)^2|$

2. $f(x) = \ln \dfrac{\sqrt{x^2+1}}{(9x-4)^2}$

3. $f(x) = \ln |\sin x|$

Natural Logarithm Function

Example

Find y'

$$3y - x^2 + \ln xy = 2$$

Natural Logarithm Function

Example

Use logarithmic differentiation to find y'.

1. $y = (4x+2)^3(6x+1)^2$
2. $y = \sqrt{4x+7}\,(x-5)^2$
3. $y = \sqrt{(3x^2+2)\sqrt{6x-7}}$

6.3 Exponential Function

Exponential Function

Example

Find $f'(x)$ where

1. $f(x) = e^{-5x}$

2. $f(x) = e^{3x^2}$

3. $f(x) = \dfrac{1}{e^x + 1}$

4. $f(x) = e^{\frac{1}{x}} + \dfrac{1}{e^x}$

Exponential Function

Example

Find y'

1. $e^{xy} - x^3 + 3y^2 = 11$
2. $e^x \cot y = xe^{2y}$

6.4 Integration(natural logarithm and exponential functions)

Integration (natural logarithm and exponential functions)

Example

Evaluate the following integrals

1. $\displaystyle\int_{-2}^{1} \frac{1}{2x+7}\,dx$

2. $\displaystyle\int_{1}^{2} \frac{4x}{x^2-9}\,dx$

3. $\displaystyle\int_{1}^{2} x^2 e^{3x^3}\,dx$

Integration (natural logarithm and exponential functions)

Example

Evaluate the following integrals

1. $\displaystyle \int \frac{x-2}{x^2 - 4x + 9}\, dx$

2. $\displaystyle \int \frac{\ln x}{x}\, dx$

3. $\displaystyle \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

Integration (natural logarithm and exponential functions)

Example

Evaluate the following integrals

1. $\displaystyle \int \frac{3\sin x}{1 + 2\cos x}\, dx$

2. $\displaystyle \int e^{\cos x} \sin x\, dx$

3. $\displaystyle \int \frac{\cos x \sin x}{\cos^2 x - 1}\, dx$

Integration (natural logarithm and exponential functions)

Example

Evaluate the following integrals

1. $\displaystyle \int \frac{\cos^2 x}{\sin x}\, dx$

6.5 General Exponential and Logarithmic Functions

General Exponential and Logarithmic Functions

Definition

For every $a > 0$, and $x \in \mathbb{R}$,

$$a^x = e^{x \ln a}$$

Theorem

$$\int a^x \, dx = \frac{1}{\ln a} \, a^x + c$$

General Exponential and Logarithmic Functions

Definition

The general logarithmic function with base a where $a > 0$ and $a \neq 1$ is denoted by $\log_a x$ and it is the inverse function of the general exponential function a^x

$$y = \log_a x \qquad \text{iff} \qquad x = a^y$$

$$\log_a x = \frac{\ln x}{\ln a}$$

General Exponential and Logarithmic Functions

Example

Find $f'(x)$

1. $f(x) = 7^x$
2. $f(x) = \log(x^4 + 3x^2 + 1)$
3. $f(x) = \log \ln x$
4. $f(x) = x^e + e^x + e^e$
5. $f(x) = x^{x^2}$

General Exponential and Logarithmic Functions

Example

Evaluate the following integrals

1. $\displaystyle \int_{-2}^{1} 7^x \, dx$

2. $\displaystyle \int \frac{2^x}{2^x + 1} \, dx$

3. $\displaystyle \int \frac{1}{x \log x} \, dx$

4. $\displaystyle \int \frac{5^{\tan x}}{\cos^2 x} \, dx$

Exam Problem

Exam problem

$$\int \frac{1}{x \ln x \ln(\ln x)} \, dx$$

6.7+ 6.8 Inverse Trigonometric Functions and Hyperbolic Functions

Integration of Inverse Trigonometric Functions

Theorem

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + c,$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}} \, dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$$

Integration of inverse hyperbolic functions

Theorem

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C, \quad a > 0$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C, \quad 0 < a < x$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C, \quad |x| < a$$

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{|x|}{a}\right) + C, \quad 0 < |x| < a$$

Hyperbolic Functions

Definition

The hyperbolic sine function ($\sinh$), and hyperbolic cosine function ($\cosh$) are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \forall x \in \mathbb{R}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \forall x \in \mathbb{R}$$

Theorem

$$\cosh^2 x - \sinh^2 x = 1$$

Derivatives of hyperbolic functions

Theorem

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Inverse Trigonometric and Hyperbolic Functions

Example

Find $f'(x)$

1. $f(x) = \sin^{-1}\sqrt{x}$
2. $f(x) = \tan^{-1}(3x - 5)$
3. $f(x) = \ln(\tan^{-1}x^2)$
4. $f(x) = 3^{\sin^{-1}(x^3)}$

Inverse Trigonometric and Hyperbolic Functions

Example

Evaluate the following integrals

1. $\displaystyle\int_0^4 \frac{1}{x^2 + 16}\,dx$

2. $\displaystyle\int_0^1 \frac{e^x}{1 + e^{2x}}\,dx$

3. $\displaystyle\int \frac{\cos x}{\sqrt{9 - \sin^2 x}}\,dx$

Inverse Trigonometric and Hyperbolic Functions

Example

Evaluate the following integrals

1. $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\,dx$

2. $\displaystyle\int \frac{e^x}{\sqrt{16-e^{2x}}}\,dx$

3. $\displaystyle\int \frac{1}{\sqrt{e^{2x}-25}}\,dx$

Inverse Trigonometric and Hyperbolic Functions

Example

Evaluate the following integrals

1. $\displaystyle \int \frac{1}{x\sqrt{1+x}}\,dx$

2. $\displaystyle \int \frac{1}{\sqrt{81+16x^2}}\,dx$

3. $\displaystyle \int \frac{1}{\sqrt{16x^2-9}}\,dx$

Inverse Trigonometric and Hyperbolic Functions

Example

Evaluate the following integrals

1. $\displaystyle\int \frac{1}{49 - 4x^2}\, dx$

2. $\displaystyle\int \frac{1}{x\sqrt{9 - x^4}}\, dx$

3. $\displaystyle\int \frac{1}{\sqrt{5 - e^{2x}}}\, dx$

Exam Problem

Exam problem

$$\int \frac{1}{\sec x \sqrt{9\sin^2 x - \sin^4 x}}\, dx$$

$$0 < x < \frac{\pi}{2}$$

Hyperbolic Functions

Example

Evaluate the following integrals

1. $\displaystyle \int x^2 \cosh x^3 \, dx$

2. $\displaystyle \int \frac{1}{\operatorname{sech} 7x} \, dx$

3. $\displaystyle \int \frac{\sinh \sqrt{x}}{\sqrt{x}} \, dx$

Hyperbolic Functions

Example

Evaluate the following integrals

$\boxed{1}$ $\displaystyle \int \sinh x \, \text{sech}^2 x \, dx$

$\boxed{2}$ $\displaystyle \int \cosh x \, \text{csch}^2 x \, dx$

Hyperbolic Functions

Example

Find $f'(x)$

1. $f(x) = \sinh^{-1} 5x$
2. $f(x) = \cosh^{-1} \sqrt{x}$
3. $f(x) = \tanh^{-1}(-4x)$
4. $f(x) = \operatorname{sech}^{-1}(x^2)$

6.9 Indeterminate Forms and L'Hopital's Rule

L'Hopital's Rule

Theorem

Suppose f and g are differentiable on an open interval (a, b) containing c, except possibly at c. If $\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at c and if $g'(x) \neq 0$ for $x \neq c$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Provided that $\lim_{x \to c} \dfrac{f'(x)}{g'(x)}$ exists or equals to ∞

L'Hopital's Rule

Example

1. $\displaystyle \lim_{x \to 5} \frac{\sqrt{x-1}-2}{x^2-25}$

2. $\displaystyle \lim_{x \to 0^+} x \ln x$

3. $\displaystyle \lim_{x \to \infty} (x^2-1)e^{-x^2}$

4. $\displaystyle \lim_{x \to \infty} (1+\frac{1}{x})^{5x}$

L'Hopital's Rule

Example

1. $\displaystyle\lim_{x \to 0^+} (e^x + 3x)^{\frac{1}{x}}$

2. $\displaystyle\lim_{x \to \infty} \left(\frac{x^2}{x-1} - \frac{x^2}{x+1} \right)$

3. $\displaystyle\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$

L'Hopital's Rule

Example

1. $\displaystyle\lim_{x\to\infty} \ \sqrt{x^4 + 5x^2 + 3} - x^2$

2. $\displaystyle\lim_{x\to\infty} \ [\ln(4x+3) - \ln(3x+4)]$

Exam Problem

Exam problem

Compute $\displaystyle\lim_{x \to +\infty} (1 + e^x)^{e^{-x}}$