Math 106
Integral Calculus
Exercises Chapter 5 (Applications of Definite Integral)

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Exercises

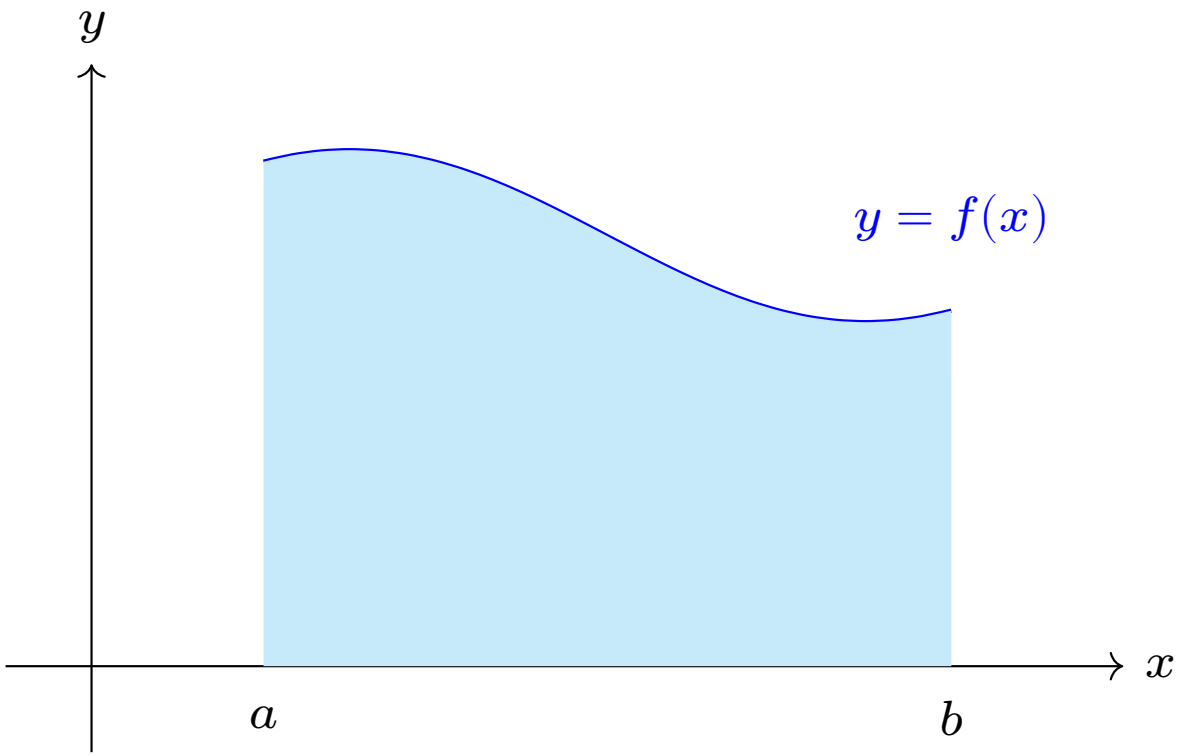
1. 5.1 Area
2. 5.2 Volume by Disk or Washer Method
3. 5.3 Volume by Cylindrical Shells
4. 5.5 Arc Length and Surface of Revolution

5.1 Area

Area

Let $f(x)$ be a continuous function on $[a, b]$, and $f(x) \geq 0$, then the area A of the region under the graph of f from a to b is given by the

$$A = \int_a^b f(x)dx$$



Area

Theorem

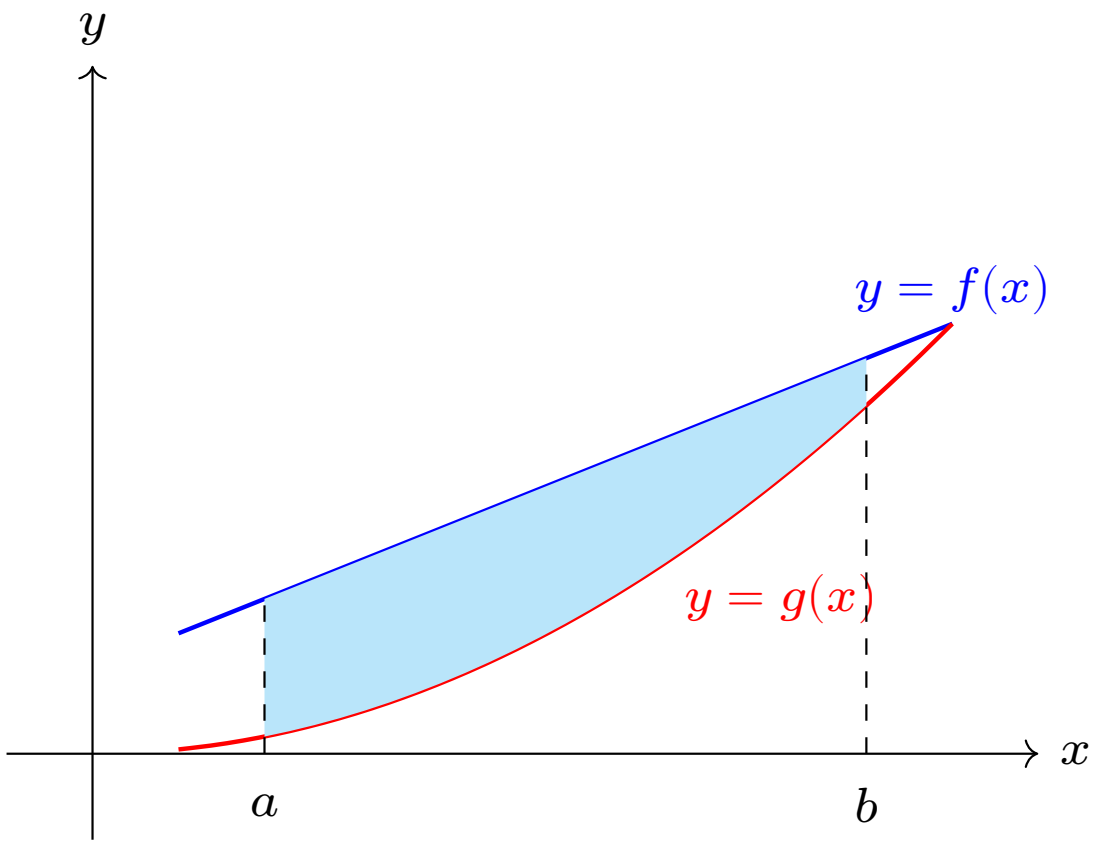
If f and g are continuous functions on $[a, b]$, and $f(x) \geq g(x)$, then the area A of the region bounded by the f and g and the vertical lines $x = a, x$ and $x = b$ is

$$\int_a^b [f(x) - g(x)]dx$$

Graphs of some functions

- $y = 2x$
- $y = -3x$
- $y = x^2$
- $y = x^3$
- $y = x + 4$
- $y = \sqrt{x}$
- $y = \sqrt[3]{x}$
- $y = (x + 1)^2$
- $y = x^2 - 2$

Area



Guidelines for finding the area

- Sketch the region and find the points of intersection of the graphs if needed
- Identify which function is on top: $f(x) \geq g(x)$.
- Subtract the lower function from the upper function.
- Integrate the difference:

$$\boxed{A = \int_a^b [f(x) - g(x)]\, dx.}$$

Area

Example

Sketch the region bounded by the graphs of the equations and find its area

$$y = x^2 \quad \text{and} \quad y = 4x$$

Area

Example

Sketch the region bounded by the graphs of the equations and find its area

$$x + y = 3 \quad \text{and} \quad y + x^2 = 3$$

Area

Example

Sketch the region bounded by the graphs of the equations and find its area

$$y = \frac{1}{x^2} \quad \text{and} \quad y = -x^2, \qquad x = 1,\ x = 2$$

Area

Example

Sketch the region bounded by the graphs of the equations and find its area

$$y = x^3 \quad \text{and} \quad y = x^2$$

Area

Example

Sketch the region bounded by the graphs of the equations and find its area

$$x = y^2 \quad \text{and} \quad y - x = 2, \qquad y = -2,\ y = 3$$

Area

Example

Sketch the region bounded by the graphs of the equations and find its area

$$x = y^2 \quad \text{and} \quad x - y = 2,$$

Area

Example

Find the are of the the region bounded by the graphs of the equations by integrating with respect to

a) x

b) y

$$y = \sqrt{x} \quad \text{and} \quad y = -x, \qquad x = 1,\ x = 4$$

Area

Example

Find the are of the the region bounded by the graphs of the equations by integrating with respect to

a) x

b) y

$$y = 1 - x^2 \quad \text{and} \quad y = x - 1$$

Area

Example

Find the are of the the region bounded by the graphs

$$f(x) = 6 - 3x^2 \quad \text{and} \quad g(x) = 3x, \quad x \in [0, 2]$$

Area

Exam problem

Sketch the region bounded by $y = x^2 + 1, y = 3x + 1$, and find its area

Area

Exam problem

Find the area of the region bounded by the curves $x = y^2$, $x + y = 6$, $y = -4$, $y = 2$

5.2 Volume by Disk or Washer Method

Volume by Disk (Revolution about $x-$axis)

Theorem

Let f be continuous on $[a, b]$ and R be the region bounded by the graph of f, the $x-$axis and the vertical lines $x = a$ and $x = b$, then the volume V of the solid of revolution generated by revolving R about the $x-$axis is

$$V = \int_a^b \pi[f(x)]^2 dx$$

Volume by Disk (Revolution about $y-$axis)

Theorem

Let g be continuous on $[c, d]$ and R be the region bounded by the graph of $x = f(y)$, the $y-$axis and the horizontal lines $y = c$ and $y = d$, then the volume V of the solid of revolution generated by revolving R about the $y-$axis is

$$V = \int_c^d \pi[g(y)]^2 dy$$

Volume by Washer

Theorem

If f and g are continuous functions such, $f(x) \geq g(x) \geq 0$ on $[a, b]$ and R be the region bounded by f, g , and $x = a$ and $x = b$, then the volume V of the solid of revolution generated by revolving R about the $x-$axis is

$$V = \int_a^b \pi[f(x)^2 - g(x)^2]dx$$

Volume by Washer

Theorem

If f and g are continuous functions, $f(y) \geq g(y) \geq 0$ on $[c, d]$ and R be the region bounded by $x = f(y)$, $x = g(y)$, and $y = c$ and $y = d$, then the volume V of the solid of revolution generated by revolving R about the $y-$axis is

$$V = \int_c^d \pi[f(y)^2 - g(y)^2]dy$$

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the x-axis

$$y = \frac{1}{x}, \quad x = 1, \quad x = 3, \quad y = 0$$

Volume

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the x-axis

$$y = \sqrt{x}, \quad x = 4, \quad y = 0$$

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the x-axis

$$y = x^3, \quad x = -2, \quad y = 0$$

Volume

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the y-axis

$$y = x^2, \quad y = 2$$

Volume

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the y-axis

$$x = y^2, \quad x - y = 2$$

Volume

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the x-axis

$$y = \sin 2x, \quad x = 0, \quad x = \pi, \quad y = 0$$

5.3 Volume by Cylindrical Shells

Volume by Cylindrical Shells

Theorem

Let f be continuous on $[a, b]$ where $a \geq 0$, and $f(x) \geq 0$, and R be the region under the graph of f between a and b, then the volume V of the solid of revolution generated by revolving R about the $y-$axis is

$$V = \int_a^b 2\pi x f(x) dx$$

Volume by Cylindrical Shells

Theorem

If g is continuous functions, on $[c, d]$ where $c \geq 0$, and $g(y) \geq 0$, and R be the region bounded by $x = g(y)$, the $y-$axis, $y = c$ and $y = d$, then the volume V of the solid of revolution generated by revolving R about the $x-$axis is

$$V = \int_c^d 2\pi y g(y) dy$$

Volume

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the y-axis

$$y = \sqrt{x}, \quad x = 4, \quad y = 0$$

Volume

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the y-axis

$$y = \frac{1}{x}, \quad x = 1, \quad x = 2, \quad y = 0$$

Volume

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the y-axis

$$y = x^2, \quad y^2 = 8x$$

Volume

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the y-axis

$$2x - y = 4, \quad x = 0, \quad y = 0$$

Volume

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the x-axis

$$x^2 = 4y, \quad y = 4$$

Volume

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the x-axis

$$y = 2x, \quad y = 6, \quad x = 0$$

Volume

Example

Sketch the region bounded by the graphs and find the volume of the solid obtained by revolving the region about the x-axis

$$y = \sqrt{x+4}, \quad y = 0, \quad x = 0$$

Exam Problem

Exam problem

Find the area of the surface obtained by revolving the curve $x = 2y^3$, $0 \leq y \leq 1$, about the y-axis

Exam Problem

Exam problem

Find the volume obtained by revolving the region bounded by $y = x^2$ and $y = \sqrt{x}$ about the x-axis

5.5 Arc Length and Surface of Revolution

Arc Length

Theorem

Let f be differentiable on $[a, b]$, and f' is continuous. The arc length of the graph of f from $A(a, f(a))$ and $B(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Arc Length

Theorem

Let $x = g(y)$ be differentiable on $[c, d]$, and g' is continuous. The arc length of the graph of g from $C(g(c), c)$ and $D(g(d), d)$ is

$$\boxed{L = \int_c^d \sqrt{1 + \left(g'(y)\right)^2}\, dy}$$

Arc Length

Example

Find the length of the arc of $y = \frac{2}{3}x^{2/3}$ between $(1, \frac{2}{3})$ and $(8, \frac{8}{3})$

Arc Length

Example

Find the length of the arc of $y = 5 - \sqrt{x^3}$ between $(1, 4)$ and $(4, -3)$

Arc Length

Example

Find the length of the arc of $30xy^3 - y^8 = 15$ between $(\frac{8}{15}, 1)$ and $(\frac{271}{240}, 2)$

Arc Length

Example

Find the length of the arc of $x = \frac{y^4}{16} + \frac{1}{2y^2}$ between $(\frac{9}{8}, -2)$ and $(\frac{9}{16}, -1)$

Arc Length

Example

Set up an integral for finding the length of the arc of $2y^3 - 7y + 2x = 8$ between $(3, 2)$ and $(4, 0)$

Arc Length

Exam problem

Compute the arc length of the curve $y = \cosh x \ , \quad 0 \leq x \leq 1$

Area of Surface of Revolution

Theorem

Let f be differentiable on $[a,b]$, and f' is continuous and $f(x) \geq 0$. The area S of the surface generated by revolving the graph of f about the $x-$axis is

$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx}$$

Area of Surface of Revolution

Theorem

Let $x = g(y)$ be differentiable on $[c, d]$, and g' is continuous and $g(y) \geq 0$. The area S of the surface generated by revolving the graph of f about the $y-$axis is

$$\boxed{S = \int_c^d 2\pi g(y)\sqrt{1 + \left(g'(y)\right)^2}\, dy}$$

Area of Surface of Revolution

Example

Find the area of the surface obtained by revolving the curve $4x = y^2$ from $(0, 0)$ to $(1, 2)$ about the x-axis

Area of Surface of Revolution

Example

Find the area of the surface obtained by revolving the curve $y = x^3$ from $(1, 1)$ to $(2, 8)$ about the x-axis

Area of Surface of Revolution

Example

Find the area of the surface obtained by revolving the curve $y = 2\sqrt{x+1}$ from $(0, 2)$ to $(3, 4)$ about the x-axis

Area of Surface of Revolution

Example

If the smaller arc of the circle $x^2 + y^2 = 25$ between $(-3, 4)$ to $(3, 4)$ is revolved find the area of the surface

a) about the x-axis

b) about the y-axis

Area of Surface of Revolution

Exam problem

Find the area of the surface obtained by revolving the curve $y = \frac{1}{2}x^2, 0 \leq x \leq 1$ about the y-axis

Area of Surface of Revolution

Exam problem

Compute the area of the surface obtained by revolving the curve $y = \sqrt{9 - x^2}$, $-2 \leq x \leq 2$, about the x-axis