1.7 Exercises

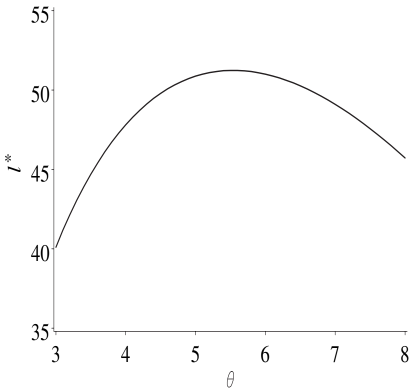

Figure 1.2 *Graph showing the location of the maximum likelihood estimate for the data in Table 1.2 on tropical cyclones.*

Table 1.3 *Successive approximations to the maximum likelihood estimate of the mean number of cyclones per season.*

k	$\theta^{(k)}$	l^*
1	5	50.878
2	6	51.007
3	5.5	51.242
4	5.75	51.192
5	5.625	51.235
6	5.5625	51.243
7	5.5313	51.24354
8	5.5469	51.24352
9	5.5391	51.24360
10	5.5352	51.24359

1.1 Let Y_1 and Y_2 be independent random variables with

$Y_1 \sim \mathrm{N}(1,3)$ and $Y_2 \sim \mathrm{N}(2,5)$. If $W_1 = Y_1 + 2Y_2$ and $W_2 = 4Y_1 - Y_2$, what is the joint distribution of W_1 and W_2?

1.2 Let Y_1 and Y_2 be independent random variables with $Y_1 \sim \mathrm{N}(0,1)$ and $Y_2 \sim \mathrm{N}(3,4)$.

(a) What is the distribution of Y_1^2?

(b) If $\mathbf{y} = \begin{bmatrix} Y_1 \\ (Y_2-3)/2 \end{bmatrix}$, obtain an expression for $\mathbf{y}^T\mathbf{y}$. What is its distribution?

(c) If $\mathbf{y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ and its distribution is $\mathbf{y} \sim \mathrm{MVN}(\boldsymbol{\mu}, \mathbf{V})$, obtain an expression for $\mathbf{y}^T\mathbf{V}^{-1}\mathbf{y}$. What is its distribution?

1.3 Let the joint distribution of Y_1 and Y_2 be MVN$(\boldsymbol{\mu}, \mathbf{V})$ with

$$\boldsymbol{\mu} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix}.$$

(a) Obtain an expression for $(\mathbf{y} - \boldsymbol{\mu})^T \mathbf{V}^{-1} (\mathbf{y} - \boldsymbol{\mu})$. What is its distribution?

(b) Obtain an expression for $\mathbf{y}^T \mathbf{V}^{-1} \mathbf{y}$. What is its distribution?

1.4 Let $Y_1, \ldots, Y_n$ be independent random variables each with the distribution $\mathrm{N}(\mu, \sigma^2)$. Let

$$\overline{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \quad \text{and} \quad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2.$$

(a) What is the distribution of $\overline{Y}$?

(b) Show that $S^2 = \frac{1}{n-1} \left[\sum_{i=1}^{n} (Y_i - \mu)^2 - n(\overline{Y} - \mu)^2 \right]$.

(c) From (b) it follows that $\sum (Y_i - \mu)^2 / \sigma^2 = (n-1) S^2 / \sigma^2 + \left[(\overline{Y} - \mu)^2 n / \sigma^2 \right]$. How does this allow you to deduce that $\overline{Y}$ and S^2 are independent?

(d) What is the distribution of $(n-1) S^2 / \sigma^2$?

(e) What is the distribution of $\frac{\overline{Y} - \mu}{S / \sqrt{n}}$?

1.5 This exercise is a continuation of the example in Section 1.6.2 in which $Y_1, \ldots, Y_n$ are independent Poisson random variables with the parameter θ.

(a) Show that $\mathrm{E}(Y_i) = \theta$ for $i = 1, \ldots, n$.

(b) Suppose $\theta = e^{\beta}$. Find the maximum likelihood estimator of β.

(c) Minimize $S = \sum \left(Y_i - e^{\beta} \right)^2$ to obtain a least squares estimator of β.

1.6 The data in Table 1.4 are the numbers of females and males in the progeny of 16 female light brown apple moths in Muswellbrook, New South Wales, Australia (from Lewis 1987).

(a) Calculate the proportion of females in each of the 16 groups of progeny.

(b) Let Y_i denote the number of females and n_i the number of progeny in each group ($i = 1, \ldots, 16$). Suppose the Y_i's are independent random variables each with the Binomial distribution

$$f(y_i; \theta) = \binom{n_i}{y_i} \theta^{y_i} (1 - \theta)^{n_i - y_i}.$$

Find the maximum likelihood estimator of θ using calculus and evaluate it for these data.

(c) Use a numerical method to estimate $\widehat{\theta}$ and compare the answer with the one from (b).

Table 1.4 *Progeny of light brown apple moths.*

Progeny group	Females	Males
1	18	11
2	31	22
3	34	27
4	33	29
5	27	24
6	33	29
7	28	25
8	23	26
9	33	38
10	12	14
11	19	23
12	25	31
13	14	20
14	4	6
15	22	34
16	7	12