Chapter 6: Univariate Random Variables
Part (3)

Continuous Distribution

uniform distribution

Q1: The current (in mA) measured in a piece of copper wire is known to follow a uniform distribution over the interval [0, 25]. Write the formula for the probability density function f(x) of the random variable X representing the current. Calculate the mean and variance of the distribution and find the cumulative distribution function F(x).

Given that $X \sim Uniform\ (a, b)$ $\quad f(x) = \frac{1}{b-a} \quad ; a \le x \le b$

$$f(x) = \frac{1}{25-0} = \frac{1}{25} \quad ; 0 \le x \le 25$$

Mean $= \frac{a+b}{2} = \frac{0+25}{2} = \frac{25}{2} = 12.5$

Variance $= \frac{(b-a)^2}{12} = \frac{(25-0)^2}{12} = 52.083$

cumulative distribution function F(x)$= \int_a^x f(t)dt = \frac{1}{b-a} \int_a^b 1dx = \frac{1}{(b-a)} [t]_a^x = \frac{x-a}{2(b-a)}, a \le x \le b$

F(x)$= \frac{x-0}{2(25-0)} = \frac{x}{50} \quad , 0 \le x \le 25$

Normal Distribution

Q2: **Resting heart rate was measured for a group of students; the students then drank 6 ounces of coffee. Ten minutes later their heart rates were measured again. The change in heart rate followed a normal distribution, with a mean increase of 5 beats per minute (μ = 5) and a standard deviation of 10 (σ = 10)**

Let *X* denotes the change in heart rate for a randomly selected person.

1) **The probability that for a randomly selected person the change in heart rate greater than 3 is**

A)	0.57926	B)	0.42074	C)	0.00135	D)	0.99868

2) **If the population size is 100000 persons, The number of people has change in heart rate between 2 to 4 is**

A)	38209	B)	7808	C)	0.07808	D)	0.38209

3) **Let P(X < K) = 0.99111 , The value of K is**

A)	2.37	B)	23.7	C)	28.7	D)	0.37

Exponential Distribution

Q3: Let X be an **exponential** random variable with parameter $\theta = ln(3)$. Compute the following probability: $P(2 \le X \le 4)$.

Solution :

$X \sim exp(\ \theta\ ) \Rightarrow \quad f(x) = \theta e^{-\theta x} \quad ; \quad F(x) = 1 - e^{-\theta x} \quad , x > 0$

$\boldsymbol{X \sim exp(\ \theta = ln(3)\ )}$

$\boldsymbol{f(x) = ln(3)\ e^{-ln(3)\,x} \ ; \ x > 0}$

$\boldsymbol{F(x) = 1 - e^{-ln(3)\,x} \quad ; x > 0}$

pdf: $P(2 \le x \le 4) = \int_2^4 f(x)dx = \ln(3) \int_2^4 e^{-\ln(3)x}\ dx = \frac{8}{81} = 0.0988$

cdf: $P(2 \le x \le 4) = p(X \le 4) - P(X \le 2) = 1 - e^{-(4)\,ln(3)} - (1 - e^{-(2)\,ln(3)}) = 0.0988$

Q4: Suppose the random variable has an **exponential** distribution with parameter $\theta = 1$. compute $P(X > 2)$.

Solution :

$X \sim exp(\ \theta\ ) \Rightarrow \quad f(x) = \theta e^{-\theta x} \quad ; \quad F(x) = 1 - e^{-\theta x} \quad , x > 0$

$\boldsymbol{x \sim exp\ (\theta = 1)}$

$f(x) = \boldsymbol{e^{-x}} \ ; \ x > 0$

$F(x) = \boldsymbol{1 - e^{-x}} \quad ; x > 0$

pdf: $P(x > 2) = \int_2^\infty f(x)dx = \int_2^\infty e^{-x}\ dx = -\ e^{-x}\ \Big|_2^\infty = -(e^{-\infty} - e^{-2}) = e^{-2} = 0.1353$

cdf: $P(x > 2) = 1 - p(x \le 2) = 1 - (1 - e^{-2}) = e^{-2}$

Q5:What is the probability that a random variable X is less than its expected value, if X has an **exponential** distribution with parameter θ?

Solution:

$X \sim \exp(\theta)$

$f(x) = \theta e^{-\theta x} \quad , x > 0 \quad ; \quad F(x) = 1 - e^{-\theta x} \quad , x > 0$

$$E(x) = \int_0^\infty \theta\, e^{-\theta x}\, dx = \theta \frac{\Gamma(2)}{\theta^2} = 1/\theta$$

$$P(x < E(X)) = P\left(x < \frac{1}{\theta}\right) = F\left(\frac{1}{\theta}\right) = 1 - e^{-\theta\left(\frac{1}{\theta}\right)} = 1 - e^{-1} = 0.6321$$

6.14.6 Properties of the exponential distribution

1. **Lack of memory property for $\mathcal{E}(\lambda)$:** For $x, y > 0$,

$$\mathbb{P}(X > x + y \mid X > x) = \mathbb{P}(X > y). \tag{6.141}$$

2. **Link to Poisson distribution:** Suppose that $X \sim \mathcal{E}(\lambda)$, and we regard X as the time between successive occurrences of some type of event (e.g., the arrival of a new insurance claim at an insurance office), where time is measured in appropriate units (seconds, minutes, hours, or days, etc.).

Now, imagine choosing a starting time (say labeled as $t = 0$), and from this point onward, we begin recording times between successive events. Let N represent the number of events (claims) that have occurred when one unit of time has elapsed. Then N will be a random variable related to the times of the occurring events. The distribution of N is Poisson with parameter λ.

3. **Minimum of independent exponential variables:** If $X_1, X_2, \ldots, X_n$ are independent exponential random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$\min(X_1, X_2, \ldots, X_n) \sim \mathcal{E}\left(\sum_{i=1}^{n} \lambda_i\right). \tag{6.142}$$

Lack of Memory Property:

Q6:Suppose a device has a lifetime X that follows an Exponential distribution with a mean of 10 hours.

assume the device has already been running for 5 hours without failing. What is the probability that it survives for at least another 10 hours?

Using Lack of Memory Property: P(X > x+y | mid X > x) = P(X > y)

P(X > 5+10| mid X > 5) = P(X > 10)

we compute: P(X > 10)

given that the mean $=E(X) = \frac{1}{\lambda}$=10 which means that $\lambda = \frac{1}{10}$

P(X > 10) = $e^{-\lambda x} = e^{-\frac{1}{10}(10)}$ =0.3679

Thus,

P(X > 15 | X > 5) = 0.3679

This shows that the probability of surviving another 10 hours does not depend on the fact that it has already survived 5 hours. The past does not affect the future, which is exactly the lack of memory property in action.

Link to the Poisson Distribution

Q7: Suppose that calls arrive at a customer service center according to an Exponential distribution with a rate of $\lambda = 4$ calls per hour.

- The time between consecutive calls follows:

$X \sim$exp(λ)= exp(4)

- The number of calls in 1 hour follows a Poisson distribution:

$N \sim$ Poisson (λt)= Poisson (4(1))

- Probability of exactly 3 calls in an hour:

P(N = 3) = $\frac{e^{-4}\ 4^3}{3!}$ = 0.1954.

The number of calls in 2 hour follows a Poisson distribution

Minimum of Independent Exponential Random Variables

Q8: Consider two independent servers processing jobs:

- Server A takes an Exponential time with λ_1= 3 jobs per hour.
- Server B takes an Exponential time with λ_2 = 5 jobs per hour.

Find the expected time until the first job is completed:

Let $X_1 \sim \exp(3)$ be the service time of Server A, and $X_2 \sim \exp(5)$ be the service time of Server B.

- The minimum processing time follows:

$\min(X_1, X_2) \sim \text{Exp}(\lambda_1+\lambda_2) = \text{Exp}(3+5) = \text{Exp}(8)$

- Expected time until the first job is completed:

$E[\min(X_1, X_2)] = \frac{1}{8} = 0.125$ hours (7.5 minutes).

Gamma Distribution

Q9: et X be a Gamma random variable with $\alpha = 4$ and $\lambda = \frac{1}{2}$. Compute $P(2 < X < 4)$?

Solution: Gamma dis : $f(x) = \frac{\lambda^{\alpha}\ e^{-\lambda x}\ x^{\alpha-1}}{\Gamma(\alpha)} = \frac{\lambda e^{-\lambda x}\ (\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \quad , x \geq 0$

$$P(2 < X < 4) = \int_2^4 \frac{\frac{1}{2}\ e^{-\frac{x}{2}}\ \left(\frac{x}{2}\right)^{4-1}}{\Gamma(4)}\ dx = \frac{1}{2^4 \Gamma(4)} \int_2^4 x^3 e^{-\frac{x}{2}}\ dx$$

$$\therefore \Gamma(4) = 3! = 6$$

By use calculate we get $\frac{1}{96} \int_2^4 x^3 e^{-\frac{x}{2}}\ dx = \mathbf{0.1239}$

Q10: (HW) if X has a probability density function given by

$$f(x) = \begin{cases} 4x^2\ e^{-2x} & ; x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and the variance?

Solution : Gamma dis : $f(x) = \frac{\lambda^{\alpha}\ e^{-\lambda x}\ x^{\alpha-1}}{\Gamma(\alpha)} = \frac{\lambda e^{-\lambda x}\ (\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \quad , x \geq 0$

$X \sim Gamma\,(\alpha = 3\,, \lambda = 2) \Rightarrow \;\; E(X) = \frac{\alpha}{\lambda} = \frac{3}{2} \;\; ; \;\; V(X) = \frac{\alpha}{\lambda^2} = \frac{3}{4}$

Or $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx \Rightarrow \quad E(X) = \int_0^{\infty} 4x^3\, e^{-2x}\, dx = 4\frac{\Gamma(4)}{2^4} = \frac{4\,(3!)}{2^4} = \frac{3}{2} = 1.5$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx \Rightarrow E(X^2) = \int_0^{\infty} 4x^4\, e^{-2x}\, dx = 4\frac{\Gamma(5)}{2^5} = \frac{4(4!)}{2^5} = 3$$

$$V(X) = E(X^2) - [E(X)]^2 = 3 - (1.5)^2 = \frac{3}{4} = 0.75$$

Q11: (HW) Let X be a gamma random variable with $\alpha = 2$ and $\lambda = 3$. Compute $P(X > 3)$?

Solution: Gamma dis : $f(x) = \frac{\lambda^{\alpha}\, e^{-\lambda x}\, x^{\alpha - 1}}{\Gamma(\alpha)} = \frac{\lambda e^{-\lambda x}\, (\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, x \geq 0$

$$f(x) = \frac{3}{\Gamma(2)}\,(3x)^{2-1}\, e^{-3x} = \frac{\mathbf{3^2}}{\mathbf{\Gamma(2)}}\,\mathbf{x\, e^{-3x}}$$

$$P(X > 3) = 1 - P(X < 3) = 1 - \left[\frac{3^2}{\Gamma(2)}\int_0^3 x\, e^{-3x}\, dx\right] = 1 - 0.9988 = \mathbf{0.001234}$$

Q12: Suppose the continuous random variable X has the following pdf:

$$f(x) = \begin{cases} \frac{1}{16} x^2\, e^{-\frac{x}{2}} & if \quad x > 0 \\ 0 & otherwise \end{cases}$$

Find $E(X^3)$?

Solution :

$$E(X^3) = \frac{1}{16}\int_0^{\infty} x^5\, e^{-\frac{x}{2}}\, dx = \frac{1}{16}\,\frac{\Gamma(6)}{\left(\frac{1}{2}\right)^6} = 480$$

$$E(X^3) = 480$$

Q13(HW) If we have

a. $f(x) = \frac{1}{b-a} \quad ; a \leq x \leq b$

b. $f(x) = \lambda e^{-\lambda x} \quad ; \; x > 0$

c. $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] \quad ; -\infty < x < \infty$

Find E(X) and V(X).

Solution :

a) $f(x) = \frac{1}{b-a} \quad ; a \le x \le b$

$$E(X) = \int_a^b x\, f(x)dx = \frac{1}{b-a}\int_a^b x\, dx = \frac{1}{(b-a)}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{2(b-a)}(b^2-a^2)$$

$$= \frac{1}{2(b-a)}(b+a)(b-a) = \frac{b+a}{2}$$

$$E(X^2) = \int_a^b x^2\, f(x)dx = \frac{1}{b-a}\int_a^b x^2\, dx = \frac{1}{(b-a)}\left[\frac{x^3}{3}\right]_a^b = \frac{(b^3-a^3)}{3(b-a)}$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{(b^2+ab+a^2)}{3}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{(b^2+ab+a^2)}{3} - \frac{(b+a)^2}{4}$$

$$= \left(\frac{4}{4}\right)\frac{1}{3}(b^2+ab+a^2) - \left(\frac{3}{3}\right)\frac{1}{4}(b^2+2ab+a^2)$$

$$= \frac{1}{12}(b^2-2ab+a^2) = \frac{1}{12}(b-a)^2$$

b) $f(x) = \lambda e^{-\lambda x} \quad ; x > 0$

$$E(X) = \int_0^\infty x\, f(x)dx = \lambda\int_0^\infty x\, e^{-\lambda x}\, dx = \lambda\,\frac{\Gamma(2)}{\lambda^2} = \frac{1}{\lambda}$$

$$\left[\text{by use} \int_0^\infty x^a\, e^{-b\,x}\, dx = \frac{\Gamma(a+1)}{b^{a+1}}, \qquad \Gamma(a) = (a-1)!\right]$$

$$E(X^2) = \int_0^\infty x^2\, f(x)dx = \lambda\int_0^\infty x^2\, e^{-\lambda x}\, dx = \lambda\frac{\Gamma(3)}{\lambda^3} = \frac{2}{\lambda^2}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

c) $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] \quad ; -\infty < x < \infty$

$$E(X) = \int_{-\infty}^\infty x\, f(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty x\, \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$

Let $u = \frac{x-\mu}{\sigma} \Rightarrow x = \sigma u + \mu \to dx = \sigma\, du \to \frac{1}{\sigma}dx = du$

$-\infty < x < \infty \quad \Rightarrow \quad -\infty < u < \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma\, u + \mu) e^{-\frac{1}{2}u^2}\, du = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u\, e^{-\frac{1}{2}u^2}\, du + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du = \mu$$

Odd function: $f(-x) = -f(x)$

Even function: $f(-x) = f(x)$

Where $\int_{-\infty}^{\infty} u\, e^{-\frac{1}{2}u^2}\, du = 0$ *because it is odd function* over a symmetric range.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}u^2}\, du = 1$$

Note: the pdf of standard normal distribution $f(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$

Then $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du = 1$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$

Let $u = \frac{x-\mu}{\sigma} \Rightarrow x = \sigma u + \mu \rightarrow dx = \sigma\, du \rightarrow \frac{1}{\sigma} dx = du$

$$-\infty < x < \infty \Rightarrow -\infty < u < \infty$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma u + \mu)^2 e^{-\frac{1}{2}u^2}\, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma^2 u^2 + 2\sigma\mu\, u + \mu^2) e^{-\frac{1}{2}u^2}\, du$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 e^{-\frac{1}{2}u^2}\, du + \frac{2\sigma\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u\, e^{-\frac{1}{2}u^2}\, du + \frac{\mu^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du = \sigma^2 + \mu^2$$

where $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 e^{-\frac{1}{2}u^2}\, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du = 1$

.$V(X) = E(X^2) - [E(X)]^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$

Note:

- $\frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 e^{-\frac{1}{2}u^2}\, du = \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} u^2 e^{-\frac{1}{2}u^2}\, du$

$$Let\ z = u^2 \rightarrow\ dz = 2u\,du \rightarrow\ \frac{1}{2u}\,dz = du\ \rightarrow \frac{1}{2\,Z^{\frac{1}{2}}}\,dz = du$$

$$0 < u < \infty \quad \Rightarrow \quad 0 < z < \infty$$

$$= \frac{2\,\sigma^2}{\sqrt{2\pi}} \int_0^\infty z\, e^{-\frac{1}{2}Z}\, \frac{1}{2Z^{\frac{1}{2}}}\, dz = \frac{\sigma^2}{\sqrt{2\pi}} \int_0^\infty Z^{\frac{1}{2}}\, e^{-\frac{1}{2}Z}\, dz = \frac{\sigma^2}{\sqrt{2\pi}}\, \frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)^{\frac{3}{2}}} = \frac{2\,\sigma^2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2}\right) = \frac{2\,\sigma^2}{\sqrt{\pi}}\,\frac{1}{2}\sqrt{\pi} = \sigma^2$$

$$we\ know\ \ \Gamma(x+1) = x\Gamma(x) \qquad and\ \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$then\ \ \Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

- $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u\, e^{-\frac{1}{2}u^2}\, du = \frac{1}{\sqrt{2\pi}} \left[-\, e^{-\frac{1}{2}u^2}\right]_{-\infty}^{\infty} = \frac{-1}{\sqrt{2\pi}}\left(e^{-\infty} - e^{-\infty}\right) = 0 \qquad \because e^{-\infty} = 0$

- $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}u^2}\, du$

$$Let\ z = u^2 \rightarrow\ dz = 2u\,du \rightarrow\ \frac{1}{2u}\,dz = du\ \rightarrow \frac{1}{2\,Z^{\frac{1}{2}}}\,dz = du$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{1}{2}\, z^{-\frac{1}{2}}\, e^{-\frac{1}{2}Z}\, dz = \frac{1}{\sqrt{2\pi}}\, \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi} = 1$$

Pareto Distribution

Q14: suppose that the wealth (in million of dollars) of individuals in certain country follows pareto with $m = 1\ million(minimum\ wealth\ is\ 1\ million) and\ \alpha = 2.5$

$$f(x) = \frac{\alpha m^\alpha}{x^{\alpha+1}}\ , x > m$$

$$f(x) = \frac{(2.5)1^{2.5}}{x^{2.5+1}} = \frac{2.5}{x^{3.5}}, x > 1 \qquad F(x) = \left(\frac{m}{x}\right)^\alpha$$

a. Find the probability that a randomly chosen individual has wealth less than 5$ million

$$P(X < 5) = \int_1^5 \frac{2.5}{x^{3.5}}\, dx = 0.98211$$

b. Find the expected wealth of an individual. $E(X) = \frac{\alpha m}{\alpha - 1} = \frac{2.5(1)}{2.5-1} = 1.667$

c. Find the variance of wealth $V(X) = \frac{\alpha m^2}{(\alpha-1)^2(\alpha-2)} = \frac{(2.5)\,1^2}{(2.5-1)^2(2.5-2)} = 2.222$

Q15: If $X \sim Exp(2)$ independent of $Y \sim Gamma(3,4)$, find:

a. E(XY).
b. $E(X^2\, Y^3)$.
c. V(X -Y)
d. V(3X + 2Y)

where

	pdf	E(X)	V(X)
$X \sim Exp(\lambda)$	$f(x) = \lambda e^{-\lambda x} \quad ; x > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$Y \sim Gamma(\alpha, \beta)$	$f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \quad ; y > 0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$

Solution :

$$X \sim Exp(\lambda = 2); \;\; Y \sim Gamma(\alpha = 3, \beta = 4)$$

a) $E(XY) = E(X)E(Y) = \frac{1}{2}\,\frac{3}{4} = \frac{3}{8}$

b) $E(X^2Y^3) = E(X^2)E(Y^3) = \frac{1}{2}\,\frac{15}{16} = \frac{15}{32}$

$V(X) = E(X^2) - [E(X)]^2 \Rightarrow \boldsymbol{E(X^2) = V(X) + [E(X)]^2} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

$E(Y^3) = \int_0^\infty \boldsymbol{y^3\, f(y) dy} = \frac{4^3}{\Gamma(3)} \int_0^\infty \boldsymbol{y^3\, y^{3-1}\, e^{-4y}\, dy}$

$$= \frac{4^3}{2} \int_0^\infty y^5\, e^{-4y}\, dy = \frac{4^3}{2}\,\frac{\Gamma(6)}{4^6} = \frac{5!}{128} = \frac{15}{16}$$

c) $\boldsymbol{V(X - Y) = V(X) + V(Y)} = \frac{1}{4} + \frac{3}{16} = \frac{7}{16}$

d) $V(3X + 2Y) = 9V(X) + 4V(Y) = 9\left(\frac{1}{4}\right) + 4\left(\frac{3}{16}\right) = 3$

Q16: Suppose X follows a standard normal distribution $N(0,1)$, meaning it has:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

We define the transformation:

$$Y = u(X) = e^X$$

To find the PDF of Y, we first determine the inverse function:

$$v(Y) = \ln(Y)$$

and compute its derivative:

$$v'(Y) = \frac{1}{Y}$$

Thus, the PDF of Y is:

$$f_Y(y) = f_X(\ln(y)) \cdot \frac{1}{y}$$

Q17: Let X be uniformly distributed between $[0,1]$, meaning:

$$f_X(x) = 1, \quad 0 \leq x \leq 1$$

Define $Y = X^2$. The CDF of Y is:

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F_X(\sqrt{y})$$

Since $F_X(x) = x$ for $X \sim U(0,1)$, we get:

$$F_Y(y) = \sqrt{y}, \quad 0 \leq y \leq 1$$

Differentiating to get the PDF:

$$f_Y(y) = \frac{1}{2\sqrt{y}}, \quad 0 \leq y \leq 1$$