Math 204
Differential Equations

Exercises
Fourier Series and Fourier Integral

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Fourier Series and Fourier Integral (Exercises)

Fourier Series

Fourier Series

Definition (Fourier Series)

If f and f' are piecewise continuous on the interval $[-L, L]$, or f is define outside the interval $[-L, L]$ so that it is periodic with period $2L$, then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

where the coefficients are given by:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

The Fourier series converges to $f(x)$ if f is continuous at x, and to

$$\frac{f(x^{+}) + f(x^{-})}{2}$$

if f is discontinuous at x.
For $x = L$ and $x = -L$, the series converges to

$$\frac{f(-L^{+}) + f(L^{-})}{2}$$

Symmetric Functions

- If $f(x)$ is even, then Fourier coefficients

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad n = 1, 2, ...$$

$$b_n = 0, \quad n = 1, 2, ...$$

- If $f(x)$ is odd, then Fourier coefficients

$$a_0 = 0, \quad n = 0, 1, 2, ...$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \quad n = 1, 2, ...$$

Fourier Cosine and Sine Series

If a function f is defined on $(0, L)$, and we want to represent this function by a trigonometric series, then we can extend $f(x)$ to an odd function f_o

$$f_o(x) = \begin{cases} f(x), & 0 < x < L, \\ -f(-x), & -L < x < 0. \end{cases}$$

with $f_o(x + 2L) = f_o(x)$ which can be approximated by $\sin$ series we can extend $f(x)$ to an even function f_e

$$f_e(x) = \begin{cases} f(x), & 0 < x < L, \\ f(-x), & -L < x < 0. \end{cases}$$

with $f_e(x + 2L) = f_e(x)$ which can be approximated by $\cos$ series

Fourier Cosine Series

Definition

Let $f(x)$ be piecewise continuous function on the interval $[0, L]$. The Fourier cosine series of f on $[0, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx, \qquad n = 0, 1, 2, ...$$

Fourier Sine Series

Definition

Let $f(x)$ be piecewise continuous function on the interval $[0, L]$. The Fourier sine series of f on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, ...$$

Fourier Series

Example

* Find the Fourier series to the 2π-periodic function

$$f(x) = \begin{cases} x, & 0 \le x \le \pi, \\ 0, & -\pi < x < 0 \end{cases}$$

Sketch on $(-3\pi, 3\pi)$, and deduce the values of

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Example

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$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$x = 0, \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}, \qquad x = \frac{\pi}{2}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

Example

* Find the Fourier cosine series of

$$f(x) = \sin x, \qquad x \in (0, \pi)$$

and deduce the values of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1}$$

Example

* Sketch 2π-periodic function f and obtain Fourier series

$$f(x) = \begin{cases} -\cos x, & -\pi < x < 0, \\ \cos x, & 0 < x < \pi \end{cases}$$

and deduce the value of

$$\sum_{n=1}^{\infty} \frac{n \sin 2n}{4n^2 - 1}$$

Example

* Sketch 2π-periodic function f and obtain Fourier series

$$f(x) = \begin{cases} -\cos x, & -\pi < x < 0, \\ \cos x, & 0 < x < \pi \end{cases}$$

and deduce the value of

$$\sum_{n=1}^{\infty} \frac{n \sin 2n}{4n^2 - 1}$$

$$f(x) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{n}{4n^2 - 1} \sin 2nx$$

$x = 1, \qquad \sum_{n=1}^{\infty} \frac{n \sin 2n}{4n^2-1} = \frac{\pi}{8} \cos 1$

Example

* If $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic odd function

$$f(x) = x(\pi - x) \qquad x \in [0, \pi]$$

Sketch the graph of f on $[-2\pi,\ 2\pi]$. Then find Fourier series of f and deduce the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$$

Hint: $\sin(2n+1)\frac{\pi}{2} = (-1)^n$

Fourier Series

Example

* If $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic odd function

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Hint: $\sin(2n+1)\frac{\pi}{2} = (-1)^n$

$$f(x) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin(2n+1)x$$

$x = \frac{\pi}{2}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$

Fourier Integral

Fourier Integral

Definition

The Fourier Integral of a function f defined on $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]\, d\lambda$$

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)\, dt$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin(\lambda t)\, dt$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t)\cos(\lambda t - \lambda x)\, dt\, d\lambda$$

Fourier Integral

Theorem

If f is absolutely integrable

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

and f, f' are piecewise continuous on every finite interval, then the Fourier integral of f converges to $f(x)$ at a point of continuity and converges to

$$\frac{f(x^+) + f(x^-)}{2}$$

at a point of discontinuity.

Fourier Sine and Cosine Integrals

If f is an odd function on $(-\infty, \infty)$, then

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)\,dt = 0$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin(\lambda t)\,dt = 2\int_{0}^{\infty} f(t)\sin(\lambda t)\,dt$$

Fourier Sine and Cosine Integrals

If f is an even function on $(-\infty, \infty)$, then

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)\,dt = 2\int_{0}^{\infty} f(t)\cos(\lambda t)\,dt$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin(\lambda t)\,dt = 0$$

Example

* Let

$$f(x) = \begin{cases} |x|, & |x| \leq 2 \\ 0, & |x| > 2 \end{cases}$$

Sketch the graph of f. Find the Fourier integral of f and deduce that

$$\int_0^\infty \frac{\sin 2\lambda}{\lambda}\, d\lambda = \int_0^\infty \frac{\sin^2 \lambda}{\lambda^2}\, d\lambda$$

Example

* Let

$$f(x) = \begin{cases} |x|, & |x| \le 2 \\ 0, & |x| > 2 \end{cases}$$

Sketch the graph of f. Find the Fourier integral of f and deduce that

$$\int_0^\infty \frac{\sin 2\lambda}{\lambda}\, d\lambda = \int_0^\infty \frac{\sin^2 \lambda}{\lambda^2}\, d\lambda$$

for $x \neq \pm 2$

$$f(x) = \frac{2}{\pi} \int_0^\infty \left[\frac{2\sin 2\lambda}{\lambda} + \frac{\cos 2\lambda - 1}{\lambda^2}\right] \cos \lambda x \, d\lambda$$

Example

* Let

$$f(x) = \begin{cases} \sin x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$$

Sketch the graph of f. Find the Fourier integral of f and deduce the value of

$$\int_0^\infty \frac{\sin^2 \lambda}{\pi^2 - \lambda^2}\, d\lambda$$

Example

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$$f(x) = \begin{cases} \sin x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$$

Sketch the graph of f. Find the Fourier integral of f and deduce the value of

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$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda\pi}{1 - \lambda^2} \sin \lambda x \, d\lambda$$

Example

* Let

$$f(x) = \begin{cases} \sin x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$$

Sketch the graph of f. Find the Fourier integral of f and deduce the value of

$$\int_0^\infty \frac{\sin^2 \lambda}{\pi^2 - \lambda^2}\, d\lambda$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda\pi}{1-\lambda^2} \sin \lambda x \, d\lambda$$

$x = \pi$, we get $\int_0^\infty \frac{\sin^2 \lambda}{\pi^2 - \lambda^2}\, d\lambda = 0$

Example

* Let

$$f(x) = \begin{cases} 0, & x < -1 \\ 1 - x, & -1 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$

Sketch the graph of f. Find the Fourier integral of f and deduce the value of

$$\int_0^\infty \frac{\sin \lambda}{\lambda}\, d\lambda$$

Example

* Let

$$f(x) = \begin{cases} 0, & x < -1 \\ 1 - x, & -1 \leq x < 1 \\ 0, & x \geq 1 \end{cases}$$

Sketch the graph of f. Find the Fourier integral of f and deduce the value of

$$\int_0^\infty \frac{\sin \lambda}{\lambda} \, d\lambda$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\frac{2 \sin \lambda}{\lambda} \cos \lambda x + \left(\frac{2 \cos \lambda}{\lambda} - \frac{2 \sin \lambda}{\lambda^2} \right) \sin \lambda x \right] d\lambda$$

Example

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$$\int_0^{\infty} \frac{\sin \lambda}{\lambda} \, d\lambda$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\frac{2 \sin \lambda}{\lambda} \cos \lambda x + \left(\frac{2 \cos \lambda}{\lambda} - \frac{2 \sin \lambda}{\lambda^2} \right) \sin \lambda x \right] d\lambda$$

$x = 0$, we get $\int_0^{\infty} \frac{\sin \lambda}{\lambda} \, d\lambda = \frac{\pi}{2}$