Chapter 6: Univariate Random Variables
Part(1)

Q1. Consider the experiment of flipping a balanced coin three times independently.

Let X= Number of heads – Number of tails.

a) List the elements of the sample space S.

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}, n = 2^3 = 8$$

b) Assign a value x of X to each sample point.

S	TTT	TTH	THT	THH	HHH	HTT	HTH	HHT
X	0-3= **-3**	1-2= **-1**	1-2= **-1**	2-1= **1**	3-0= **3**	1-2= **-1**	2-1=**1**	2-1=**1**

c) Find the probability distribution function of X.

x	-3	-1	1	3	total
f(x)	1/8	3/8	3/8	1/8	1

d) Find $P(X \leq 1) = F(1) = \left(\frac{1}{8}\right) + \left(\frac{3}{8}\right) + \left(\frac{3}{8}\right) = \frac{7}{8}$

e) Find $P(X < 1) = \left(\frac{1}{8}\right) + \left(\frac{3}{8}\right) = \frac{4}{8} = \frac{1}{2}$

f) Find $\mu = E(X)$

$$E(X) = \sum_x x.f(x) = -3\left(\frac{1}{8}\right) - 1\left(\frac{3}{8}\right) + 1\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = 0$$

g) Find $s^2 = Var(X)$

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = \sum_x x^2.f(x) - \left(E(X)\right)^2$$

$$= 9\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 1\left(\frac{3}{8}\right) + 9\left(\frac{1}{8}\right) - 0 = 3$$

Q2. Let X be a random variable with the following probability distribution:

x	-3	6	9

f(x)	0.1	0.5	0.4

a) Find the mean (expected value) of X , $\mu = E(X)$

$$E(X) = \sum_x x.f(x) = -3(0.1) + 6(0.5) + 9(0.4) = 6.3$$

b) Find $E(X^2)$

$$E(X^2) = \sum_x x^2.f(x) = (-3)^2\,(0.1) + 6^2(0.5) + 9^2(0.4) = 51.3$$

c) Find the variance of X, $Var\,(X) = \sigma_x^2$

$$\sigma_x^2 = E(X^2) - \left(E(X)\right)^2 = 51.3 - (6.3)^2 = 11.61$$

d) Find the mean of 2X+1, $E(2X+1) = \mu_{2X+1}$

$$= 2E(X) + E(1) = 2(6.3) + 1 = 13.6$$

e) Find the variance of 2X+1, $Var(2X+1) = \sigma_{2\,X+1}^2$

$$= 2^2\,Var(X) + Var(1) = 4(11.61) + 0 = 46.44$$

Q3. Which of the following is a probability distribution function:

(a) $f\,(x) = \frac{x+1}{10}$; $x = 0,1,2,3,4$

$f(0) = \left(\frac{1}{10}\right) = 0.1 < 1$; $f(1) = \left(\frac{2}{10}\right) = 0.2 < 1$; $f(2) = \left(\frac{3}{10}\right) = 0.3 < 1$;
$f(3) = \left(\frac{4}{10}\right) = 0.4 < 1$; $f(4) = \left(\frac{5}{10}\right) = 0.5 < 1$

$$\sum f(x) = \frac{1+2+3+4+5}{10} = 1.5 \neq 1 \quad \therefore f(x)\ is\ not\ PDF$$

(b) $f\,(x) = \frac{x-1}{5}$; $x = 0,1,2,3,4$

$f(0) = \frac{-1}{5} < 0 \quad \therefore f(x)\ is\ not\ PDF.$

(c) $f\,(x) = \frac{1}{5}$; $x = 0,1,2,3,4$

$f(0) = f(1) = f(2) = f(3) = f(4) = \frac{1}{5}$

$\sum f(x) = \frac{1+1+1+1+1}{5} = 1 \qquad \therefore f(x)\ is\ PDF$

(d) $f\,(x) = \frac{5-x^2}{6}$; $x = 0,1,2,3$

$f(0) = \frac{5}{6} < 1$; $f(1) = \frac{4}{6} < 1$; $f(2) = \frac{1}{6} < 1$; $f(3) = -\frac{4}{6} < 0$

$since\ f(3) < 0\ ,\ f(x)\ is\ not\ PDF$

Q4. Let X be a discrete random variable with the probability distribution function:

$$f(x) = kx \quad for\ x = 1, 2, and\ 3.$$

(i) Find the value of k. $we\ know\ that\ \sum_x f(x) = 1$

$$\sum_x kx = 1 \rightarrow k + 2k + 3K = 1 \rightarrow 6k = 1 \rightarrow k = 1/6$$

$$\boldsymbol{f(x) = \frac{x}{6}\ ; x = 1, 2, 3}$$

(ii) Find the cumulative distribution function (CDF), F(x)

$F(1) = P(X \leq 1) = P(X = 1) = f(1) = 1/6$

$F(2) = P(X \leq 2) = P(X = 1) + P(X = 2) = 3/6$

$F(3) = P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3) = 1$

$$F(x) = P(X \leq x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \leq x < 2 \\ 3/6 & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

(iii) Using the CDF, F(x), find P (0.5 $<$ $X \leq$ 2.5).

$P\ (0.5 < X \leq 2.5) = P(X \leq 2.5) - P(X \leq 0.5)$

$$= F(2.5) - F(0.5) = \left(\frac{3}{6}\right) - 0 = \frac{3}{6} = \frac{1}{2}$$

$Or\ by\ use\ \ f(x){:}\ P\ (0.5 < X \leq 2.5) = f(1) + f(2) = \frac{1}{6} + \frac{2}{6} = \frac{1}{2}$

Q5. Let X be a random variable with cumulative distribution function (CDF) given by:

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.25 & 0 \leq x < 1 \\ 0.6 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

(a) Find the probability distribution function of X, $f(x)$.

$$f(x) = F(x) - F(x-1)$$

$f(0) = 0.25 - 0 = 0.25$

$f(1) = 0.6 - 0.25 = 0.35$

$f(2) = 1 - 0.6 = 0.4$

$$f(x) = \begin{cases} 0.25 & x = 0 \\ 0.35 & x = 1 \\ 0.4 & x = 2 \\ 0 & otherwise \end{cases}$$

(b) Find $P(1 \leq X < 2)$. (using both $f(x)$ and $F(x)$)

By using $f(x)$:

$P(1 \leq X < 2) = P(X = 1) = f(1) = 0.35$

By using *F(X)* :

$P(1 \leq X < 2) = F(2-1) - F(1-1) = F(1) - F(0) = 0.6 - 0.25 = 0.35$

(c) Find $P(X > 2)$. (using both f(x) and F(x))

By using $f(x)$:

$P(X > 2) = 1 - P(X \leq 2) = 1 - [f(0) + f(1) + f(2)] = 0$

By using *F(X)* : $P(X>2)=1-F(2)=1-1=0$

Find $P(1<X\le 2)=F(2)-F(1)=1-0.6=0.4$

Find $P(1\le X\le 2)=F(2)-F(1-1)=1-0.25=0.75$

Find $P(1<X<2)=F(2-1)-F(1)=F(1)-F(1)=0$

Note: For a discrete random variable

$$P(a\le X<b)=F(b-1)-F(a-1)$$

$$P(a<X\le b)=F(b)-F(a)$$

$$P(a\le X\le b)=F(b)-F(a-1)$$

$$P(a<X<b)=F(b-1)-F(a)$$

Q6.The r.v. X has pdf $f(x)=\begin{cases}C(1-x^2), & -1<x<1\\ 0, & elswhere\end{cases}$

a. What is the value of **C**.
b. Find the following probabilities using the *pdf* of X:
   i. $P(X<0)$
   ii. $P(X\ge\frac{1}{2})$
   iii. $P(-\frac{1}{2}<X\le\frac{1}{2})$
   iv. $P(X>1)$
c. What is the *cdf* of X.
d. Find the probabilities in (b) using the *cdf*.
e. Survival function of X
f. The hazard Rate.

Solution :

a) We know $\int_{-\infty}^{\infty} f(x)\,dx=1 \rightarrow \int_{-1}^{1} c(1-x^2)\,dx=1$

$$\rightarrow c\left[x-\frac{x^3}{3}\right]_{-1}^{1}=1 \rightarrow c\left[1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right]=1 \rightarrow c\left[2-\frac{2}{3}\right]=1$$

$$\rightarrow \frac{4}{3}c=1 \rightarrow \boldsymbol{c=\frac{3}{4}=0.75}$$

$$\therefore f(x)=\frac{3}{4}(1-x^2) \quad , \quad -1\le x\le 1$$

b) Using pdf

i. $P(x<0)=\frac{3}{4}\int_{-1}^{0}(1-x^2)\,dx=\frac{3}{4}\left[x-\frac{x^3}{3}\right]_{-1}^{0}=\frac{3}{4}\left[0-\left(-1+\frac{1}{3}\right)\right]=\frac{1}{2}$

ii. $P\left(x \geq \frac{1}{2}\right) = \frac{3}{4}\int_{0.5}^{1}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{0.5}^{1} = \frac{3}{4}\left[\left(1-\frac{1}{3}\right) - \left(\frac{1}{2}-\frac{1}{24}\right)\right] = \frac{5}{32}$

iii. $P\left(-\frac{1}{2} \leq x < \frac{1}{2}\right) = \frac{3}{4}\int_{-0.5}^{0.5}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-0.5}^{0.5}$

$$= \frac{3}{4}\left[\left(\frac{1}{2}-\frac{1}{24}\right) - \left(-\frac{1}{2}+\frac{1}{24}\right)\right] = \frac{11}{16}$$

iv. $P(X > 1) = 0$

c) $F(x) = \begin{cases} 0 & , \quad x < -1 \\ \frac{3}{4}\int_{-1}^{x}(1-t^2)\,dt = \frac{3}{4}\left[t - \frac{t^3}{3}\right]_{-1}^{x} = \boldsymbol{\frac{3}{4}x - \frac{1}{4}x^3 + \frac{1}{2}} & , \quad -1 \leq x < 1 \\ 1 & , \quad 1 \leq x \end{cases}$

d) Using cdf

i. $P(x < 0) = F(0) = \frac{3}{4}(0) - \frac{1}{4}(0)^3 + \frac{1}{2} = \frac{1}{2}$

ii. $P\left(x \geq \frac{1}{2}\right) = 1 - P\left(X < \frac{1}{2}\right) = 1 - F\left(\frac{1}{2}\right) = 1 - \left[\frac{3}{4}\left(\frac{1}{2}\right) - \frac{1}{4}\left(\frac{1}{2}\right)^3 + \frac{1}{2}\right]$

$$= 1 - \frac{27}{32} = \frac{5}{32}$$

iii. $P\left(-\frac{1}{2} \leq x < \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) = \frac{27}{32} - \left[\frac{3}{4}\left(\frac{-1}{2}\right) - \frac{1}{4}\left(\frac{-1}{2}\right)^3 + \frac{1}{2}\right] = \frac{11}{16}$

iv. $P(X > 1) = 1 - P(X \leq 1) = 1 - F(1) = 1 - 1 = 0$

e) Survival function of $S_X(x) = 1 - F(x) = 1 - \left(\frac{3}{4}x - \frac{1}{4}x^3 + \frac{1}{2}\right) = \boldsymbol{\frac{1}{2} - \frac{3}{4}x + \frac{1}{4}x^3}$

f) The hazard Rate. $h(x) = \frac{S_X(x)}{f(x)} = \frac{\frac{1}{2}-\frac{3}{4}x+\frac{1}{4}x^3}{\frac{3}{4}(1-x^2)}$

Q8. Let X be a continuous random variable on (0,1) with probability density function

$$f(x) = \begin{cases} 3x^2, & 0 \leq x \leq 1 \\ 0, & elswhere \end{cases}$$

And let the random variable Y

$$Y = \begin{cases} X, & 0 \leq X \leq \frac{1}{3} \\ \frac{1}{3}, & X > \frac{1}{3} \end{cases}$$

Compute $P(Y = \frac{1}{3})$

Since $Y = \frac{1}{3}$ $occure\ when\ X > \frac{1}{3}$ then

$$P\left(Y = \frac{1}{3}\right) = P\left(X > \frac{1}{3}\right)$$

$$= \int_{\frac{1}{3}}^{1} 3x^2\,dx$$

$$= \left[3\frac{x^3}{3}\right]_{\frac{1}{3}}^{1} = [x^3]_{\frac{1}{3}}^{1} = 1 - \frac{1}{27} = \frac{26}{27}$$

Q8.A system can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = Cxe^{-x/2} \quad ; x > 0$$

a. What is the probability that the system functions for at least 5 months.
b. What is the probability that the system functions from 3 to 6 months.
c. What is the probability that the system functions less than 1 month.

Solution :

a) To find C, we know that $C\int_0^{\infty} x\, e^{-\frac{x}{2}}\ dx = 1$

$$[\, use \int_0^{\infty} x^a\, e^{-b\,x}\ dx = \frac{\Gamma(a+1)}{b^{a+1}}\ , \qquad \Gamma(a) = (a-1)!\,]$$

$$C\,\frac{\Gamma(2)}{\left(\frac{1}{2}\right)^2} = 1 \Rightarrow \qquad C\,\frac{1}{1/4} = 1\ \Rightarrow \qquad 4C = 1 \Rightarrow \qquad \boldsymbol{C = \frac{1}{4}}$$

$$P(X \geq 5) = \frac{1}{4}\int_5^{\infty} x\, e^{-\frac{x}{2}}\ dx$$

By use calculator

$$P(X \geq 5) = 1 - P(X < 5) = 1 - \frac{1}{4}\int_0^{5} x\, e^{-\frac{x}{2}}\ dx = 1 - \frac{2.8508}{4} = 0.2873$$

Or by use Integration by Parts $\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du$

$$P(X \geq 5) = \frac{1}{4}\int_5^{\infty} x\, e^{-\frac{x}{2}}\ dx$$

Let $u = x \ \rightarrow \quad du = dx$

$dv = e^{-\left(\frac{x}{2}\right)}\, dx \ \rightarrow \quad v = -2\, e^{-\frac{x}{2}}$

$$= \frac{1}{4}\left[\left[-2x\, e^{-\frac{x}{2}}\right]_5^{\infty} + 2\int_5^{\infty} e^{-\frac{x}{2}}\ dx\ \right] = \frac{1}{4}\left[\left(0 + 10e^{-\frac{5}{2}}\right) + 2\,(-2)\int_5^{\infty} -\frac{1}{2}\, e^{-\frac{x}{2}}\ dx\right]$$

$$= \frac{1}{4}\left[10e^{-\frac{5}{2}} - 4\left[e^{-\frac{x}{2}}\right]_5^{\infty}\right] = \frac{1}{4}\left[10e^{-\frac{5}{2}} + 4e^{-\frac{5}{2}}\right] = \frac{1}{4}\left[14e^{-\frac{5}{2}}\right] = 0.2873$$

b) By use calculator $P(3 < X < 6) = \frac{1}{4}\int_3^{6} x\, e^{-\frac{x}{2}}\ dx = 0.3587$

Or by use Integration by Parts $\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du$

Let $u = x \ \rightarrow \quad du = dx \quad ; \quad dv = e^{-\left(\frac{x}{2}\right)}\, dx \ \rightarrow \quad v = -2\, e^{-\frac{x}{2}}$

$$= \frac{1}{4}\left[\left[-2x\, e^{-\frac{x}{2}}\right]_3^6 + 2\int_3^6 e^{-\frac{x}{2}}\, dx\right] = \frac{1}{4}\left[\left(-12e^{-\frac{6}{2}} + 6e^{-\frac{3}{2}}\right) + 2\,(-2)\int_3^6 -\frac{1}{2}\, e^{-\frac{x}{2}}\, dx\right]$$

$$\frac{1}{4}\left[\left(-12e^{-3} + 6e^{-\frac{3}{2}}\right) - 4\left[e^{-\frac{x}{2}}\right]_3^6\right] = \frac{1}{4}\left[-12e^{-3} + 6e^{-\frac{3}{2}} - 4e^{-3} + 4e^{-\frac{3}{2}}\right]$$

$$= \frac{1}{4}\left[10e^{-\frac{3}{2}} - 16e^{-3}\right] = \frac{1}{4}(1.435)$$

c) By use calculator $P(X < 1) = \frac{1}{4}\int_0^1 x\, e^{-\frac{x}{2}}\, dx = 0.0902$

Or by use Integration by Parts $\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du$

Let $u = x \rightarrow \quad du = dx$

$dv = e^{-\left(\frac{x}{2}\right)}\, dx \rightarrow \quad v = -2\, e^{-\frac{x}{2}}$

$$= \frac{1}{4}\left[\left[-2x\, e^{-\frac{x}{2}}\right]_0^1 + 2\int_0^1 e^{-\frac{x}{2}}\, dx\right] = \frac{1}{4}\left[\left(-2e^{-\frac{1}{2}}\right) + 2\,(-2)\int_0^1 -\frac{1}{2}\, e^{-\frac{x}{2}}\, dx\right]$$

$$= \frac{1}{4}\left[-2e^{-\frac{1}{2}} - 4\left[e^{-\frac{x}{2}}\right]_0^1\right] = \frac{1}{4}\left[-2e^{-\frac{1}{2}} - 4e^{-\frac{1}{2}} + 4\right] = \frac{0.3608}{4} = 0.0902$$

Q9. The cumulative distribution function of a continuous r.v. Y is given by

$$F(y) = \begin{cases} 0\ , & y \le 3 \\ 1 - \dfrac{9}{y^2}\ , & y > 3 \end{cases}$$

Find

a. $P(Y \le 5)$.

b. $P(Y > 8)$.

c. The *pdf* of Y

Solution :

a) $P(Y \le 5) = F(5) = 1 - \frac{9}{25} = \frac{16}{25} = 0.64$

b) $P(Y > 8) = 1 - P(Y \le 8) = 1 - F(8) = 1 - \left(1 - \frac{9}{64}\right) = \frac{9}{64} = 0.1406$

c) $f(y) \underset{derivative}{\overset{integration}{\rightleftarrows}} F(y)$

$$f(y) = \frac{d}{dy}\, F(y) = F(y)' = \frac{18}{y^3}$$

$$\therefore\ f(x) = \begin{cases} \dfrac{18}{y^3}\ , & y > 3 \\ 0\ , & 0.w \end{cases}$$

Q10. If the density function of the continuous r.v. X is $f(x) = \begin{cases} x \quad , & 0 < x < 1 \\ 2 - x \quad , & 1 \le x < C \\ 0 \;, & 0.w \end{cases}$

Find

a. The value of C.
b. The cumulative distribution function of X.
c. $P(0.8 < X < 0.6\,C)$.

Solution :

a) $\int_{-\infty}^{\infty} f(x)\;\; dx\; = 1 \;\rightarrow\;\; \int_0^1 x\;\; dx\; +\; \int_1^C (2-x)\;\; dx\; = 1$

$$\left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^C = 1 \rightarrow \frac{1}{2} + \left(2C - \frac{C^2}{2}\right) - \left(2 - \frac{1}{2}\right) = 1 \;\rightarrow\; \frac{1}{2} + 2C - \frac{C^2}{2} - \frac{3}{2} = 1$$

$$\rightarrow\; 2C - \frac{C^2}{2} - 1 = 1 \;\;\rightarrow\; 2C - \frac{C^2}{2} - 2 = 0 \;\;\rightarrow\; -\frac{C^2}{2} + 2C - 2 = 0$$

by use

$a = -\frac{1}{2}\,, b = 2\,, c = -2$

$C = \dfrac{-b\; \pm \sqrt{b^2\; - 4ac}}{2a}$

$= \frac{-2\,\pm\sqrt{4-4}}{2(-\frac{1}{2})} = -\frac{2}{-1} = 2$

Or

$-\frac{1}{2}[C^2 - 4C + 4] = 0$

$C^2 - 4C + 4 = 0$

$(C-2)^2 = 0$

$C - 2 = 0 \;\rightarrow C = 2$

$$\therefore f(x) = \begin{cases} x \quad , & 0 < x < 1 \\ 2 - x \quad , & 1 \le x < 2 \\ 0 \;, & 0.w \end{cases}$$

b) $F(x) = P(X \le x)$

$$F(x) \;\; = \begin{cases} 0 & , \; x < 0 \\ \int_0^x x\; dx = \left[\frac{1}{2}x^2\right]_0^x = \boldsymbol{\frac{1}{2}x^2} & , \; 0 \le x < 1 \\ \int_0^1 x\; dx + \int_1^x (2-x)\; dx = \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x = \boldsymbol{2x - \frac{x^2}{2} - 1} & , 1 \le x < 2 \\ 1 & , \;\; 2 \le x \end{cases}$$

c) $P(0.8 < X < 0.6\,C) = P(0.8 < X < 1.2)$ $\quad , wher\ 0.6C = 0.6(2) = 1.2$

0 0.8 1 1.2

By use *pdf*

$$= \int_{0.8}^{1} x\,dx \;+\; \int_{1}^{1.2} (2-x)dx$$

$$= \left[\frac{x^2}{2}\right]_{0.8}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{1.2} = 0.36$$

By use *cdf*

$$P(X < 1.2) - P(X < 0.8)$$

$$= F(1.2) - F(0.8)$$

$$= \left[2(1.2) - \frac{1.2^2}{2} - 1\right] - \left[\frac{1}{2}(0.8^2)\right]$$

$$= 0.68 - 0.32 = 0.36$$

NOTE: For a continuous random variable:

- $P(X \ge a) = P(X > a)$ [This NOT true for discrete r.v]
- $f(y) \underset{derivative}{\overset{integration}{\rightleftarrows}} F(y) \Rightarrow \begin{cases} f(x) = \frac{d}{dx} F(x) \\ F(x) = \int_{-\infty}^{x} f(x)\,dx \end{cases}$
- Compute probabilities using cdf:

  $P(a < X < b) = P(X \le b) - P(X \le a) = F(b) - F(a)$
- $\int_a^b e^{c\,x}\,dx = \left[\frac{1}{c} e^{c\,x}\right]_a^b \,, c \in \mathbb{R}$
- $e^{-\infty} = 0\,;\quad e^0 = 1;\quad e^{\infty} = \infty$
- $Y = \frac{u}{v}\,, v \ne 0 \;\Rightarrow\; \frac{dY}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- $\int_0^{\infty} x^a\,e^{-b\,x}\,dx = \frac{\Gamma(a+1)}{b^{a+1}}\;;\; a, b \in \mathbb{R}\;;$

$\Gamma(a) = (a-1)!\quad;\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\quad;\quad \Gamma(a+1) = a\,\Gamma(a) \to \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\sqrt{\pi}$

Q11. Consider the random variable X with the following probability distribution function:

X	0	1	2	3
f(x)	0.4	*c*	0.3	0.1

The value of ***C*** is

(A) 0.125 (B) 0.2 (C) 0.1 (D) 0.125 (E) - 0.2

we know that $\sum_x f(x) = 1$

$0.4 + C + 0.3 + 0.1 = 1 \quad \rightarrow \quad 0.8 + C = 1 \ \rightarrow \ C = 1 - 0.8 = 0.2$

Q12. compute the expected value and the standard deviation for the probability function of X given

X	-4	0	3
$f(x) = P(X = x)$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{4}{7}$

Solution :

- $E(X) = \sum_x xf(x) = (-4)\left(\frac{1}{7}\right) + (0)\left(\frac{2}{7}\right) + (3)\left(\frac{4}{7}\right) = \frac{8}{7}$

- $V(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \sum_x x^2 f(x) = (-4)^2\left(\frac{1}{7}\right) + (0)^2\left(\frac{2}{7}\right) + (3)^2\left(\frac{4}{7}\right) = \frac{52}{7}$$

$then \ \ V(X) = E(X^2) - [E(X)]^2 = \frac{52}{7} - \left(\frac{8}{7}\right)^2 = \frac{300}{49} = 6.122$

$$\sigma_X = \sqrt{{\sigma_X}^2} = \sqrt{V(X)} = \sqrt{6.122} = 2.474$$

Q13. Roll one die and let X be the resulting number. Find the expected value of Ξ:

Solution :

The possible value of X are 1,2,3,4,5 and 6

$$E(X) = \sum_x xf(x) = (1)\left(\frac{1}{6}\right) + (2)\left(\frac{1}{6}\right) + (3)\left(\frac{1}{6}\right) + (4)\left(\frac{1}{6}\right) + (5)\left(\frac{1}{6}\right) + (6)\left(\frac{1}{6}\right) = \frac{21}{6} = \frac{7}{2}$$

Q14. Let X_1, X_2 and X_3 be independent r.v.'s with means 4, 9, 3 and variances 3, 7, 5 respectively. For $Y = 2X_1 - 3\,X_2 + 4\,X_3$ and $Z = X_1 + 2\,X_2 - \ X_3$, find:

a. E(Y) and E(Z).

b. V(Y) and V(Z).

Solution :

$E(X_1) = 4\,, Var(X_1) = 3$

$E(X_2) = 9\,, Var(X_2) = 7$

$E(X_3) = 3\,, Var(X_3) = 5$

a) $E(Y) = E(2X_1 - 3X_2 + 4X_3) = 2E(X_1) - 3E(X_2) + 4E(X_3)$

$$= 2(4) - 3(9) + 4(3) = 8 - 27 + 12 = -7$$

$E(Z) = E(X_1 + 2X_2 - X_3) = E(X_1) + 2E(X_2) - E(X_3)$

$$= 4 + 2(9) - 3 = 4 + 18 - 3 = 19$$

b) Since X_1, X_2 *and* X_3 are independent

$$V(Y) = V(2X_1 - 3X_2 + 4X_3) = 4\,V(X_1) + 9\,V(X_2) + 16\,V(X_3)$$

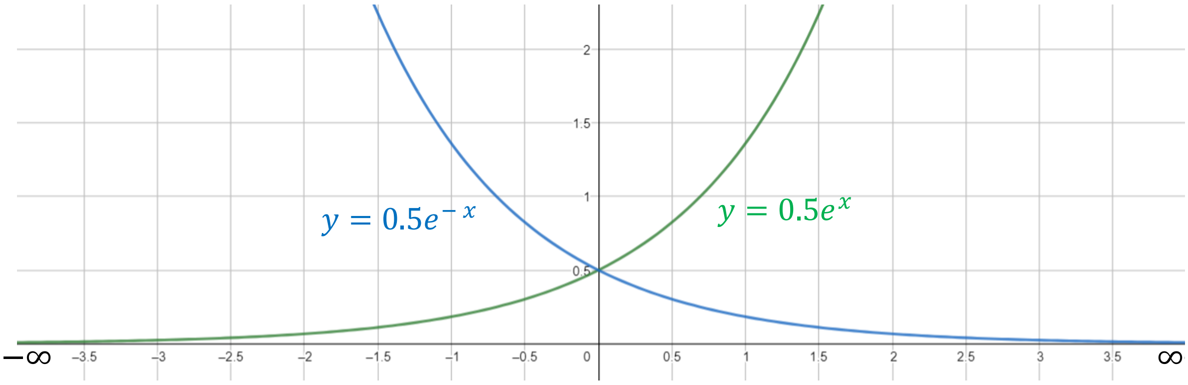
$$= 4(3) + 9(7) + 16(5) = 12 + 63 + 80 = 155$$

$$V(Z) = V(X_1 + 2X_2 - X_3) = V(X_1) + 4V(X_2) + V(X_3)$$

$$= 3 + 4(7) + 5 = 3 + 28 + 5 = 36$$

Q15. A r.v. has $f(x) = \frac{1}{2}\, e^{-|x|}\ ;\ for\ -\infty < x < \infty$, find E(X) and V(X).

Solution :



we know the definition of absolute value is $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

$$\Rightarrow \quad f(x) = \begin{cases} \frac{1}{2}e^{-x}\ , & x > 0 \\ \frac{1}{2}e^{x}\ , & x < 0 \end{cases}$$

$$E(X) = \frac{1}{2}\left[\int_{-\infty}^{0} xe^{x}\, dx + \int_{0}^{\infty} xe^{-x}\ dx\right] = \frac{1}{2}[K_1 + K_2] = \frac{1}{2}[-1 + 1] = 0$$

K_1: *substitution* $x = -z \Rightarrow dx = -dz\ ;\ -\infty < x < 0$

new limits of integration *when* $x = 0 \Rightarrow z = 0$

$$when\ x = -\infty \Rightarrow z = \infty$$

$$K_1 = \int_{\infty}^{0}(-z)e^{-z}(-dz) = -\int_{0}^{\infty} z\, e^{-z}\, dz = \frac{\Gamma(2)}{1^2} = \mathbf{-1}$$

K_2: $\int_0^\infty xe^{-x}\ dx = \Gamma(2) = 1$

Or by use Integration by Parts $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$

Let $u = x \ \rightarrow \ du = dx \ , \ dv = e^{-x}\ dx \ \rightarrow \ v = -e^{-x}$

$K_1 = [x\,e^{-x}]_0^\infty + \int_0^\infty e^{-x}\,dx = \mathbf{1}$

$\boldsymbol{V(X) = E(X^2) - [E(X)]^2}$

$$E(X^2) = \frac{1}{2}\left[\int_{-\infty}^{0} x^2 e^x\,dx + \int_0^\infty x^2 e^{-x}\ dx\right] = \frac{1}{2}[K_3 + K_4] = \frac{1}{2}[2+2] = 2$$

K_3: $by\ substitution\ x = -z \ \rightarrow \ dx = -dz \ ; \ -\infty < x < 0 \Rightarrow \ 0 < z < \infty$

$K_3 = \int_0^\infty (z^2)e^{-z}\,dz = \frac{\Gamma(3)}{1^3} = 2$

$K_4 = \int_0^\infty x^2 e^{-x}\ dx = \Gamma(3) = 2$

$$\therefore V(X) = 2 - 0 = 2$$

Q16. Let X be a discrete random variable with pmf given by the following table

x	1	2	3	4	5
p(x)	0.15	0.20	0.40	0.15	0.10

Find $Mx(t)$

Solution :

$$M_X(t) = E(e^{xt}) = \sum_x e^{xt} f(x)$$

$M_X(t) = 0.15e^t + 0.20e^{2t} + 0.40e^{3t} + 0.15e^{4t} + 0.10e^{5t}$

Q17. Let X be a continuous random variable with $f(x) = \frac{1}{b-a},\ a < x < b$, Find $M_X(t)$

Solution :

$$M_X(t) = E(e^{xt}) = \int_a^b e^{xt} f(x)dx$$

$$= \int_a^b \frac{e^{xt}}{b-a}dx$$

$$= \frac{1}{b-a}\int_a^b e^{xt}dx$$

$$= \frac{1}{b-a}\left[\frac{e^{xt}}{t}\right]\begin{matrix} b \\ a \end{matrix} = \frac{1}{b-a}\left[\frac{e^{bt}-e^{at}}{t}\right] = \frac{e^{bt}-e^{at}}{t(b-a)}$$

Q18. Let X be a discrete random variable with MGF $M_X(t) = e^{\theta(e^t-1)}$, $\theta > 0$

Find the expected value and the variance of X using moment generating functions.

$$M'_X(t) = \theta e^t e^{\theta(e^t-1)}$$

$$E(X) = M'_X(0) = \theta$$

$$M''_X(t) = \theta e^t . e^{\theta(e^t-1)} + \theta e^t . e^{\theta(e^t-1)} . \theta e^t$$

$$= \theta e^t . e^{\theta(e^t-1)} + \theta^2 e^{2t} . e^{\theta(e^t-1)}$$

$$E(X^2) = M''_X(0) = \theta + \theta^2$$

$$V(X) = E(X^2) - [E(X)]^2 = \theta + \theta^2 - \theta^2 = \theta$$

Note that $V(X) = E(X - E(X))^2 = E(X^2) - [E(X)]^2$

Q19. Let X and Y be two independent random variables with MGF.

$M_X(t) = e^{t^2-2t}$ and $M_Y(t) = e^{3t^2-2t}$

(a) Find the MGF of X+2Y

(b) Find the MGF of X+5

Solution :

(a)

$M_{X+2Y}(t) = E\left(e^{(X+2Y)t}\right)$

$= E(e^{Xt+2Yt})$

$= E(e^{Xt})\, E(e^{2Yt})$

$= E(e^{Xt})\, E\left(e^{Y(2t)}\right)$

$= M_X(t) M_Y(2t) = e^{t^2-2t} e^{3(2t)^2-2(2t)}$

$= e^{t^2-2t+12t^2-4t} = = e^{13t^2-6t}$

(b) $M_{X+5}(t) = E\left(e^{(X+5)t}\right) = e^{5t} E(e^{Xt}) = e^{5t} M_X(t) = e^{5t}\, e^{t^2-2t} = e^{t^2+3t}$

Q20. suppose X is the IQ of a random person. We assume $X \geq 0$; $E(X) = 100$; and $\sigma = 10$:

Find an upper bound of $P(X \geq 200)$ using Chebyshev's inequality.

$P(X \geq 200) = P(X - 100 \geq 200 - 100)$ Chebyshev's inequality $P(|X-\mu| > r) \leq \frac{\sigma^2}{r^2}$

$= P(X - 100 \geq 100) \leq P(|X - 100| > 100) \leq \frac{10^2}{100^2} = \frac{1}{100}$

Q21. let $\mu = 33$, and $\sigma^2 = 16$

Find the upper bound of $P(|X - 33| > 14)$

Q22. A hot dog stand has <u>mean</u> daily sales of $420 with a standard deviation of $50. The income has a <u>normal distribution</u>. What is the standardized value for daily sales of $520?

Standardized value= Z=$\frac{X-\mu}{\sigma} = \frac{520-420}{50} = 2$

Q23. Let X be a continuous random variable with $f(x) = \frac{1}{b-a},\ a < x < b,$

Find the median of X

To find the median M we solve $\int_{-\infty}^{M} f(x)\,dx = 0.5$

$$\int_a^M \frac{1}{b-a} dx = 0.5$$

$$\frac{1}{b-a}[x]_a^M = 0.5$$

$$\frac{M-a}{b-a} = 0.5$$

$$M - a = 0.5b - 0.5a$$

$$M = \frac{a+b}{2}$$

Q24. Let X be a continuous random variable with $f(x)$ =1/2 ,$1 < x < 3,$

Find the 75.5p th $percentile$ of X

$$\int_{-\infty}^{p} f(x)\,dx = 0.755$$

$$\int_1^p \frac{1}{2}dx = 0.755$$

$$\frac{1}{2}[x]_1^p = 0.755$$

$$\frac{p-1}{2}=0.755$$

$$p - 1 = 1.51$$

$$p = 2.51$$

Q25.Let X be the discrete random variable with pmf given by

$$p(x) = \begin{cases} (\frac{1}{2})^x , & x = 1,2,3,\ldots \\ 0 , & otherwise \end{cases}$$

Find the mode of X:

The value of x that maximizes $p(x)$ is $x = 1$: Thus, the mode of X is 1

Q26. Let X be the continuous random variable with pdf given by $f(x) = 0.75(1 - x^2)$ for $-1 \leq x \leq 1$ and 0 otherwise. Find the mode of X:

by using the first derivative: $f'(x) = 0$

$$-0.75(2)x = 0$$

$$x = 0$$

The pdf is maximum for $x = 0$: Thus, the mode of X is 0

Q27.Let X be any random variable . let Y=3X and then $\mu_Y = 3\mu_x$ find Cov(X,Y)

Cov(X,Y) = $E\big((X - \mu_x)(Y - \mu_Y)\big) = E\big((X - \mu_x)(3X - 3\mu_X)\big) =$

$$= 3E(X - \mu_x)^2 = 3Var(X)$$

Q28. Let $X and\ Y$ r.v.'s with $V(X) = 6.267$ $, V(Y) = 2.167$ $and\ Cov(X,Y) = 0.067$

Find $V(X + 3Y) = V(X) + 3^2V(Y) + 2(3)Cov(X,Y)$

$$= 6.267 + 9(2.167) + 6(0.067) = 26.172$$

Q29: (**mixed distribtions)**

Example 4.14

Let X be a continuous random variable with the following PDF:

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let also

$$Y = g(X) = \begin{cases} X & 0 \leq X \leq \frac{1}{2} \\ \frac{1}{2} & X > \frac{1}{2} \end{cases}$$

Find the CDF of Y.

Solution

First we note that $R_X = [0, 1]$. For $x \in [0, 1]$, $0 \leq g(x) \leq \frac{1}{2}$. Thus, $R_Y = [0, \frac{1}{2}]$, and therefore

$$F_Y(y) = 0, \qquad \text{for } y < 0,$$

$$F_Y(y) = 1, \qquad \text{for } y > \frac{1}{2}.$$

Now note that

$$\begin{aligned} P\left(Y = \tfrac{1}{2}\right) &= P\left(X > \tfrac{1}{2}\right) \\ &= \int_{\frac{1}{2}}^{1} 2x dx = \tfrac{3}{4}. \end{aligned}$$

Also, for $0 < y < \frac{1}{2}$,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X \leq y) \\ &= \int_0^y 2x dx \\ &= y^2. \end{aligned}$$

Thus, the CDF of Y is given by

$$F_Y(y) = \begin{cases} 1 & y \geq \frac{1}{2} \\ y^2 & 0 \leq y < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Q30:

Mixed Distributions:

Example 4.15

Let Y be the mixed random variable defined in Example 4.14.

a. Find $P(\frac{1}{4} \leq Y \leq \frac{3}{8})$.
b. Find $P(Y \geq \frac{1}{4})$.
c. Find EY.

Solution

Since we have the CDF of Y, we can find the probability that Y is in any given interval. We should pay special attention if the interval includes any jump points.

a. Find $P(\frac{1}{4} \leq Y \leq \frac{3}{8})$: We can write

$$\begin{aligned} P\left(\tfrac{1}{4} \leq Y \leq \tfrac{3}{8}\right) &= F_Y\left(\tfrac{3}{8}\right) - F_Y\left(\tfrac{1}{4}\right) + P\left(Y = \tfrac{1}{4}\right) \\ &= \left(\tfrac{3}{8}\right)^2 - \left(\tfrac{1}{4}\right)^2 + 0 = \tfrac{5}{64}. \end{aligned}$$

b. Find $P(Y \geq \frac{1}{4})$: We have

$$\begin{aligned} P\left(Y \geq \tfrac{1}{4}\right) &= 1 - F_Y\left(\tfrac{1}{4}\right) + P\left(Y = \tfrac{1}{4}\right) \\ &= 1 - \left(\tfrac{1}{4}\right)^2 = \tfrac{15}{16}. \end{aligned}$$

c. Find EY: Here, we can differentiate the continuous part of the CDF to obtain

$$c(y) = \frac{dC(y)}{dy} = \begin{cases} 2y & 0 \leq y \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

So, we can find EY as

$$\begin{aligned} EY &= \int_0^{\frac{1}{2}} y(2y)dy + \tfrac{1}{2}P\left(Y = \tfrac{1}{2}\right) \\ &= \tfrac{1}{12} + \tfrac{3}{8} = \tfrac{11}{24}. \end{aligned}$$

Q31:

Symmetry:

Compute the symmetry of distribution of X about $c = 0, 2$:

$$f(x) = x^2 - 2$$

Since, $f(2+x) = (2+x)^2 - 2 = 4 + 4x + x^2 - 2 \neq f(2-x) = (2-x)^2 - 2 = 4 - 4x + x^2 - 2$, then, the distribution of X is not symmeteric about 2.

Since, $f(x) = x^2 - 2 = f(-x) = (-x)^2 - 2 = x^2 - 2$, then, the distribution of X is symmeteric about 0.

Q32:

Jensen's Inequality:

$$f(x) = 2x, \qquad 0 < x < 1$$

Apply Jensen's inequality for $E(X^2)$ and $E(\sqrt{X})$:

$$E(X) = \int_0^1 x f(x) dx = \int_0^1 2x^2 dx = \frac{2}{3} \Rightarrow (E(X))^2 = \frac{4}{9} = 0.44$$

$$E(X^2) = \int_0^1 x^2 f(x) dx = \int_0^1 2x^3 dx = \frac{1}{2} = 0.5$$

Since x^2 is a convex function, then $E(X^2) > (E(X))^2$.

$$E(X) = \frac{2}{3} \Rightarrow \sqrt{E(X)} = \sqrt{\frac{2}{3}} = 0.816$$

$$E(\sqrt{X}) = \int_0^1 \sqrt{x} f(x) dx = \int_0^1 2\, x^{3/2}\, dx = \frac{4}{5} = 0.8$$

Since $\sqrt{X}$ is a concave function, then $E(\sqrt{X}) < \sqrt{E(X)}$.

Note:

For any distribution has μ *and* σ^2. Then $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ has

$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{n}{n}E(X) = \mu$

$V(\bar{X}) = \frac{1}{n^2}V(\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) = \frac{n}{n^2}V(X) = \frac{1}{n}V(X) = \frac{1}{n}\sigma^2$

Note:

- $(\boldsymbol{b} \pm \boldsymbol{a})^2 = (\boldsymbol{a}^2 \pm 2\boldsymbol{ab} + \boldsymbol{b}^2)$
- $(\boldsymbol{b}^2 - \boldsymbol{a}^2) = (\boldsymbol{b} + \boldsymbol{a})(\boldsymbol{b} - \boldsymbol{a})$
- $(\boldsymbol{b}^3 - \boldsymbol{a}^3) = (\boldsymbol{b} - \boldsymbol{a})(\boldsymbol{b}^2 + \boldsymbol{ab} + \boldsymbol{a}^2)$

- $f(x)$ even function $\Leftrightarrow f(-x) = f(x) \Leftrightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
- $f(x)$ odd function $\Leftrightarrow f(-x) = -f(x) \Leftrightarrow \int_{-a}^{a} f(x)\,dx = 0$

Extra exercises

Q1.It is known that 20% of the people in a certain human population are female. The experiment is to select a committee consisting of two individuals at random. Let X be a random variable giving the number of females in the committee.

1. List the elements of the sample space S.
2. Assign a value x of X to each sample point.
3. Find the probability distribution function of X.
4. Find the probability that there will be at least one female in the committee.
5. Find the probability that there will be at most one female in the committee.
6. Find $\mu = E(X)$
7. Find $\sigma^2 = Var(X)$

Solution :

H.W

Q2. A box contains 100 cards; 40 of which are labelled with the number 5 and the other cards are labelled with the number 10. Two cards were selected randomly with replacement and the number appeared on each card was observed. Let X be a random variable giving the total sum of the two numbers.

a) List the elements of the sample space S.

b) To each element of S assign a value x of X.
c) Find the probability mass function (probability distribution function) of X.
d) Find P(X=0).
e) Find P(X>10).
f) Find $\mu = E(X)$.
g) Find $\sigma^2 = Var(X)$.

Solution :

H.W

Q3. If the continuous random variable X has mean μ=16 and variance σ^2=5, then *P*(*X* = 16) is

(A) 0.0625 (B) 0.5 (C) 0.0 (D) None of these.

Solution :

$P(X = 16) = 0$ (Because X is Continuous r.v)

Q4. Consider the probability density function:

$$f(x) = \begin{cases} k\sqrt{x}, & 0 < x < 1 \\ 0, & elswhere \end{cases}$$

1) The value of k is:

(A)1 (B) 0.5 (C) 1.5 (D) 0.667

2) The probability $P(0.3 < X \le 0.6)$

(A)0.4647 (B) 0.3004 (C) 0.1643 (D) 0.4500

3) The expected value of X, E(X) is,

(A)0.6 (B) 1.5 (C) 1 (D) 0.667

[Hint: $\int \sqrt{x}\, dx = \frac{x^{\frac{3}{2}}}{3/2} + c$]

Solution :

1) We know $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_0^1 k\sqrt{x}\ dx = 1 \ \rightarrow \ k\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = 1 \ \rightarrow \ \frac{2}{3}k = 1 \ \rightarrow \ k = \frac{3}{2} = 1.5$$

$$\therefore\ f(x) = \frac{3}{2}\sqrt{x}\ , \quad 0 < x < 1$$

2) $P(0.3 \le X \le 0.6) = \int_{0.3}^{0.6} \frac{3}{2}\sqrt{x}\ dx = \frac{3}{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_{0.3}^{0.6} = (0.6)^{3/2} - (0.3)^{\frac{3}{2}} = 0.3004$

3) $E(x) = \int_{-\infty}^{\infty} xf(x)\, dx = \int_0^1 \frac{3}{2}\, x\sqrt{x}\ dx = \int_0^1 \frac{3}{2}\, x^{\frac{3}{2}}\ dx = \frac{3}{2}\left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 = 0.5999 \approx 0.6$

Q5. If the cumulative distribution function of the random variable X having the form:

$$P(X \le x) = F(x) = \begin{cases} 0\,, & x < 0 \\ x/(x+1)\,, & x \ge 0 \end{cases}$$

Then

1) $P(0 < X < 2)$ equals to
   (a) 0.555 (b) 0.333 (c) 0.667 (d) none of these.
2) If $P(X \le k) = 0.5$, then k equals to
   (a) 5 (b) 0.5 (c) 1 (d) 1.5
3) The pdf is
   (a) $x/(x+1)$ (b) x (c) $1/(x+1)^2$ (d) $x/(x+1)^2$

Solution :

1) $P(0 < X < 2) = F(2) - F(0) = \frac{2}{2+1} - \frac{0}{0+1} = \frac{2}{3} = 0.667$
2) $P(X \le k) = 0.5 \quad \to F(k) = 0.5 \ \to \frac{k}{k+1} = 0.5 \ \to k = 0.5k + 0.5$
   $\to k(1-0.5) = 0.5 \ \to 0.5k = 0.5 \ \to \boldsymbol{k = 1}$
3) $f(x) = \frac{d}{dx} F(x)$
   $f(x) = \frac{d}{dx}\left[\frac{x}{x+1}\right] = \frac{(x+1)(1)-x\,(1)}{(x+1)^2} = \frac{1}{(x+1\,)^2} \qquad ,x \ge 0$

Q6.Suppose continuous r.v. X has density function $f(x) = \begin{cases} C\,x^2\,, & 1 < x < 2 \\ 0, & elswhere \end{cases}$

a. Find the value of the constant C.

b. Find $P(X \ge \frac{3}{2})$.

c. Find the cumulative distribution function of X.

d. Find $P(X \ge \frac{3}{2})$ using the cdf.

Solution :

a) We know $\int_{-\infty}^{\infty} f(x)\,dx = 1 \ \to \ \int_1^2 cx^2\,dx = 1 \ \to C\left[\frac{x^3}{3}\right]_1^2 = 1$

$\to C\left(\frac{8}{3} - \frac{1}{3}\right) = 1 \ \to \ \frac{7}{3}C = 1 \ \to \ C = \frac{3}{7} \qquad \therefore f(x) = \frac{3}{7}x^2\,, \quad 1 < x < 2$

b) $P\left(X \ge \frac{3}{2}\right) = \frac{3}{7}\int_{3/2}^{2} x^2\,dx = \frac{3}{7}\left[\frac{x^3}{3}\right]_{3/2}^{2} = \frac{1}{7}\left(8 - \frac{27}{8}\right) = \frac{37}{56}$

c) $F(x) = \begin{cases} 0 & , \quad x < 1 \\ \frac{3}{7}\int_1^x t^2\,dt = \frac{3}{7}\left[\frac{t^3}{3}\right]_1^x = \frac{1}{7}(x^3 - 1) = \frac{x^3}{7} - \frac{1}{7} & , \quad 1 \le x < 2 \\ 1 & , \quad 2 \le x \end{cases}$

d) $P\left(X \ge \frac{3}{2}\right) = 1 - p\left(x \le \frac{3}{2}\right) = 1 - F\left(\frac{3}{2}\right) = 1 - \left(\frac{1}{7}\,\frac{27}{8} - \frac{1}{7}\right) = \frac{37}{56}$

Q7. The probability distribution for company **A** is given by:

x	1	2	3
f(x)	0.3	0.4	0.3

and for company **B** is given by:

y	0	1	2	3	4
f(y)	0.2	0.1	0.3	0.3	0.1

Show that the variance of the probability distribution for company B is greater than that of company A.

Company A:

$E(X) = \sum_x x.f(x) = 1(0.3) + 2(0.4) + 3(0.3) = 2$

$E(X^2) = \sum_x x^2.f(x) = 1^2(0.3) + 2^2(0.4) + 3^2(0.3) = 4.6$

$Var(X) = E(X^2) - \left(E(X)\right)^2 = 4.6 - 2^2 = \mathbf{0.6}$

Company B:

$E(Y) = 0(0.2) + 1(0.1) + 2(0.3) + 3(0.3) + 4(0.1) = 2$

$E(Y^2) = 0^2(0.2) + 1^2(0.1) + 2^2(0.3) + 3^2(0.3) + 4^2(0.1) = 5.6$

$Var(Y) = E(Y^2) - \left(E(Y)\right)^2 = 5.6 - 2^2 = \mathbf{1.6}$

$\therefore\ var(Y) > var(X)$

Q8. If X and Y are independent r.v.'s with E(X)=3, E(Y)=5, V(X)=2, and V(Y)=5, find:

a. E(XY)

b. $E(X^2Y)$

Solution :

X and Y are independent

a) $E(XY) = E(X)E(Y) = 3(5) = 15$

b) $E(X^2Y) = E(X^2)E(Y) = 11(5) = 55$

Where $V(X) = E(X^2) - [E(X)]^2 \iff 2 = E(X^2) - 3^2 \iff E(X^2) = 2 + 9 = 11$

Q9. Let X and Y are independent r.v's with p.d.f $f(x) = e^{-x};\ x > 0,$

$f(y) = e^{-y};\ y > 0$, find :

a. E(X) and V(X).
b. E(Y) and V(Y).
c. E(XY).
d. $E(X^2\, Y^3)$.
e. Cumulative function of X
f. Survival function of X
g. The hazard Rate.

Solution :

a) $\boldsymbol{E(X)} = \int_0^\infty x\, e^{-x}\ dx = \frac{\Gamma(2)}{1^2} = 1$ $\ [by\ use \int_0^\infty x^a\, e^{-b\,x}\ dx = \frac{\Gamma(a+1)}{b^{a+1}}\ , \Gamma(a) = (a-1)!\ ]$

Or by use Integration by Parts $\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du$

Let $u = x\ \rightarrow\ du = dx\ \ ,\ \ dv = e^{-x}\ dx\ \rightarrow\ v = -\,e^{-x}$

$E(X) = \int_0^\infty x\, e^{-x}\ dx = [-x\, e^{-x}]_0^\infty + \int_0^\infty e^{-x}\, dx = [\,-e^{-x}]_0^\infty = 1$

$\boldsymbol{E(X^2)} = \int_0^\infty x^2\, e^{-x}\, dx = \frac{\Gamma(3)}{1^3} = 2$

or use integration by parts

$\Rightarrow Let\ u = x^2 \rightarrow\ du = 2x\, dx\ ,\ \ dv = e^{-x}\ dx \rightarrow\ \ v = -\,e^{-x}$

$E(X^2) = [-x^2 e^{-x}]_0^\infty + 2\int_0^\infty xe^{-x}\, dx =\ 0 + 2E(X) = 2(1) = 2$

$\boldsymbol{V(X)} = E(X^2) - [E(X)]^2 = 2 - 1 = 1$

b) Same solution in part (a) , $E(Y) = 1\ \ ,V(Y) = 2$

c) As X and Y are independent $\Rightarrow E(XY) = E(X)E(Y) = 1$

d) As X and Y are independent $\Rightarrow E(X^2Y^3) = E(X^2)E(Y^3) = 2(6) = 12$
$where\ E(Y^3) = \int_0^\infty Y^3\, e^{-y}\, dy = \frac{\Gamma(4)}{1^4} = 6$

or use integration by parts

$\Rightarrow Let\ u = y^3 \rightarrow\ du = 3y^2\, dy\ ,\ \ dv = e^{-y}\ dy \rightarrow\ \ v = -\,e^{-y}$

$E(Y^3) = [-y^3 e^{-y}]_0^\infty + 3\int_0^\infty y^2 e^{-y}\, dy =\ 0 + 3E(Y^2) = 3(2) = 6$