Exercise -2-

Example: Using the graphical method, solve each of the following LPP

1- Max $Z = 10X_1 + 8X_2$

Subject to

$2X_1 + X_2 \leq 40$

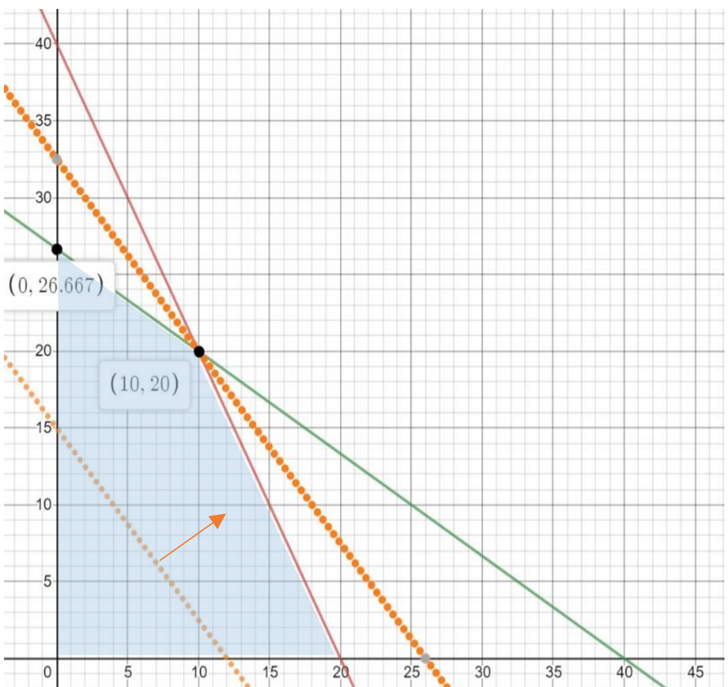
$2X_1 + 3X_2 \leq 80$

$X_1 \geq 0, X_2 \geq 0$

To determine tow point on each straight line

$2X_1 + X_2 = 40$ >> (0,40) and (20,0)

$2X_1 + 3X_2 = 80$ >> (0,26.667) and (40,0)



Z=120
(0,15), (12,0)

To determine the direction of solution region for each constraints:
Let us choose point (0,0)

Constraint 1:

$$2X_1 + X_2 \leq 40$$
$$0 \leq 40$$

Constraint 2:

$$2X_1 + 3X_2 \leq 80$$
$$0 \leq 80$$

To plot objective function line, which pass through pint (0,15)

$$10X_1 + 8X_2 = 120$$

We need other point $X_2 = 0$

$$10(X_1) = 120$$
$$X_1 = 12$$

https://www.desmos.com/calculator/ouuvx55dut

(X_1, X_2)	Z
(0,26.667)	213.33
(10,20)	260
(20,0)	200

2- Max $Z = 300X_1 + 400X_2$

Subject to

$5X_1 + 4X_2 \leq 200$

$3X_1 + 5X_2 \leq 150$

$5X_1 + 4X_2 \geq 100$

$8X_1 + 4X_2 \geq 80$

$X_1 \geq 0, X_2 \geq 0$

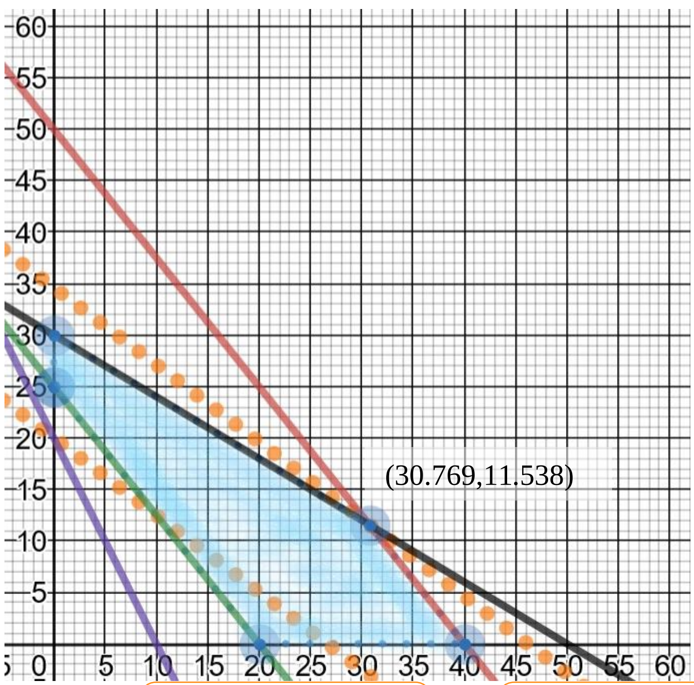
To plot graph of each constraint, we need to determine tow points.

$5X_1 + 4X_2 = 200$ >> (0,50) and (40,0)

$3X_1 + 5X_2 = 150$ >> (0,30) and (50,0)

$5X_1 + 4X_2 = 100$ >> (0,25) and (20,0)

$8X_1 + 4X_2 = 80$ >> (0,20) and (10,0)



Z=8000
(20,5), (0,20)

Z=13846
(0,34.615), (46.153,0)

https://www.desmos.com/calculator/s5s19doxbt

To determine the direction of solution region for each constraints:

Constraint 1:
Let us choose point (25،10)

$$5X_1 + 4X_2 \leq 200$$
$$165 \leq 200$$

Constraint 2:
Let us choose point (10،10)

$$3X_1 + 5X_2 \leq 150$$
$$80 \leq 150$$

Constraint 3:
Let us choose point (15،15)

$$5X_1 + 4X_2 \geq 100$$
$$135 \geq 100$$

Constraint 4:
Let us choose point (10,10)

$$8X_1 + 4X_2 \geq 80$$
$$120 \geq 80$$

To plot objective function line, which pass through pint (20,5)

$$300(X_1) + 400(X_2) = 8000$$

We need other point $X_2 = 0$

$$300(X_1) = 8000$$
$$X_1 = 26.667$$
$$(26.7,0), (0,20)$$

The values of x_1, x_2 associated with the optimum point are determined by solving the equations 1 and 2, that is,

$$(3)^* \; 5x_1 + 4x_2 = 200$$
$$(-5)^* \; \underline{3x_1 + 5x_2 = 150}$$
$$(12 - 25)x_2 = 600 - 750$$
$$-13x_2 = -150$$
$$x_2 = 11.538$$
$$5x_1 + 4(11.538) = 200$$
$$x_1 = \frac{200 - 4(11.538)}{5} = 30.769$$

Or we can find the optimal solution by compute the value of the objective function at each vertex (Extreme) pints, as follows

(X_1, X_2)	Z
(0,25)	10000
(0,30)	12000
(30.769,11.538)	13846
(40,0)	6000
(20,0)	12000

We have unique optimal solution at $x_1^* = 30.769$, $x_2^* = 11.538$ with an optimal value

Z=13846

Q: Determine the optimality condition that will keep the optimum unchanged. Also, Determine the optimality range for C1 and C2, assuming that the other coefficient is kept constant at its present value.?!

The optimal solution occurs at (30.769,11.538) intersection between (1) and (2) constraints.

$$(1)\; 5X_1 + 4X_2 \le 200$$
$$(2)\; 3X_1 + 5X_2 \le 150$$
$$\text{objective function: } Z = 300X_1 + 400X_2$$

The slop of $z = c_1x_1 + c_2x_2$ is $-\frac{c_1}{c2}$

$$\frac{-5}{4} \le \frac{-C_1}{C_2} \le \frac{-3}{5} \quad >> \quad -1.25 \le \frac{-C_1}{C_2} \le -0.6 \quad >> \quad 0.6 \le \frac{C_1}{C_2} \le 1.25.$$

suppose that coefficient C_2 is fixed at its current value of C_2=400, then the optimality range for C_1 is

$$\frac{3}{5} \leq \frac{C_1}{400} \leq \frac{5}{4}$$

$$240 \leq C_1 \leq 500$$

suppose that coefficient C_1 is fixed at its current value of C_1=300, then the optimality range for C_2 is

$$\frac{3}{5} \leq \frac{300}{C_2} \leq \frac{5}{4}$$

$$\frac{4}{5} \leq \frac{C_2}{300} \leq \frac{5}{3}$$

$$240 \leq C_2 \leq 500$$

the parameters (input data) of the model can change within certain units without causing the optimum solution to change (sensitivity analysis- change in the objective coefficients).

Q: If the objective function change to $Z = 350\, X_1 + 300\, X_2$, will the current optimal solution remain the same? What is the value of the objective function?

To check $\frac{C_1}{C_2} = \frac{350}{300} = 1.167 \quad \in [0.6, 1.25]$

Thus, the optimal solution will remain the same. The optimal profit will change to $350\,(30.769) + 300\,(11.538) = 14230.55\$$

3- Min $Z = 20X_1 + 40X_2$

Subject to

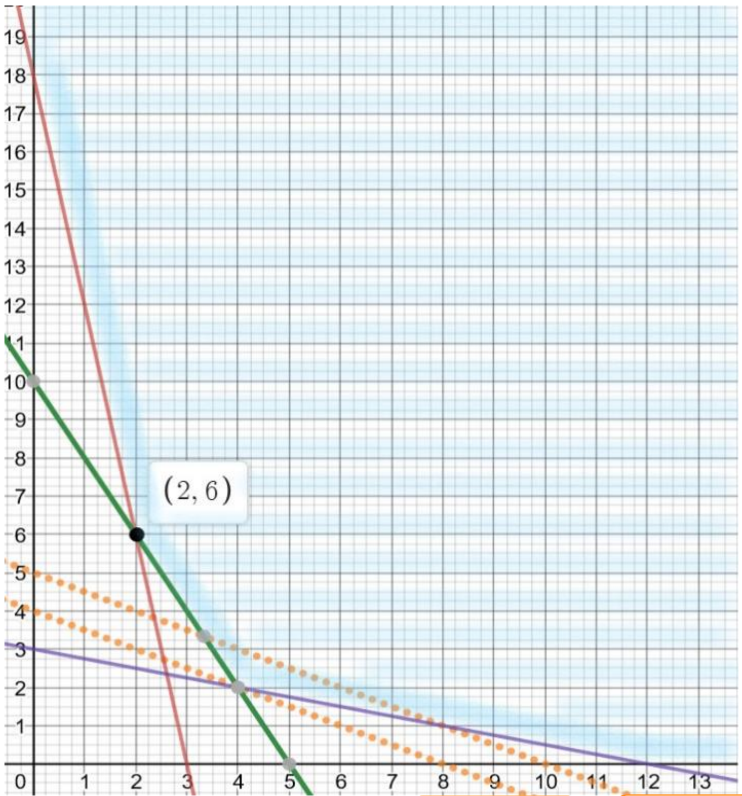
(1) $36X_1 + 6X_2 \geq 108$
(2) $3X_1 + 12X_2 \geq 36$
(3) $200X_1 + 100X_2 \geq 1000$
(4) $X_1 \geq 0, X_2 \geq 0$

Answer:

$36X_1 + 6X_2 = 108$ >> (0,18) and (3,0)

$3X_1 + 12X_2 = 36$ >> (0,3) and (12,0)

$200X_1 + 100X_2 = 1000$ >> (0,10) and (5,0)



Z=160
(8,0), (0,4)

Z=200
(0,5), (0,10)

To determine the direction of solution region for each constraints:
Let us choose point (4,4)

Constraint 1:
$$36X_1 + 6X_2 \geq 108$$
$$168 \ \ \geq 108$$
Constraint 2:
$$3X_1 + 12X_2 \geq 36$$
$$60 \geq 36$$
Constraint 3:
$$200X_1 + 100X_2 \geq 1000$$
$$1200 \geq 1000$$

To plot objective function line, which pass through pint (4,3)
$$20X_1 + 40X_2 = 200$$

We need other point $X_2 = 0$
$$20(X_1) = 200$$
$$X_1 = 10$$
(0,5), (10,0)

https://www.desmos.com/calculator/rpeajfihwx

To find optimal solution (Extreme point), by solving the equations 2 and 3

(200)* $3x_1 + 12x_2 = 36$
(-3)* $\underline{200x_1 + 100x_2 = \ 1000}$

$$(2400 - 300)x_2 = 7200 - 3000$$
$$2100x_2 = 4200$$
$$x_2 = 2$$
$$3x_1 + 12(2) = 36$$
$$x_1 = 4$$

Or we can find the optimal solution by compute the value of the objective function at each vertex (Extreme) pints, as follows

(X_1, X_2)	Z
(0,18)	720
(2,6)	280
(4,2)	160
(12,0)	240

We have unique optimal solution at $x_1^* = 4$, $x_2^* = 2$ with an optimal value Z=160

Q: Determine the optimality condition that will keep the optimum unchanged. Also, Determine the optimality rang for C1 and C2, assuming that the other coefficient is kept constant at its present value.?!

The binding constraints are:

(2) $3X_1 + 12X_2 \leq 36$

(3) $200X_1 + 100X_2 \geq 1000$

Range of optimality

$\frac{-200}{100} \leq \frac{-C_1}{C_2} \leq \frac{-3}{12}$ >> $\mathbf{0.25 \leq \frac{C_1}{C_2} \leq 2}$

$0.25 \leq \frac{20}{C_2} \leq 2$ >> $0.5 \leq \frac{C_2}{20} \leq 4$ >> $\mathbf{10 \leq C_2 \leq 80}$

$0.25 \leq \frac{C_1}{40} \leq 2$ >> $\mathbf{10 \leq C_1 \leq 80}$

4- Max $Z = 50X_1 + 18X_2$ **(H.W)**

Subject to

$2X_1 + X_2 \leq 100$

$X_1 + X_2 \leq 80$

$X_1 \geq 0, X_2 \geq 0$

5- Min $Z = 120X_1 + 100X_2$ **(H.W)**

Subject to

$10X_1 + 5X_2 \leq 80$

$6X_1 + 6X_2 \leq 66$

$4X_1 + 8X_2 \geq 24$

$5X_1 + 6X_2 \leq 90$

$X_1 \geq 0, X_2 \geq 0$

Special cases in the Graphical Method:
1) Unbounded solution.
2) Infeasible/ No solution.
3) Multiple Optimal solution.

Example: Using the graphical method, solve each of the following LPP:

1-Unbounded Solution

1- Max $Z = -2X_1 + 6X_2$

Subject to

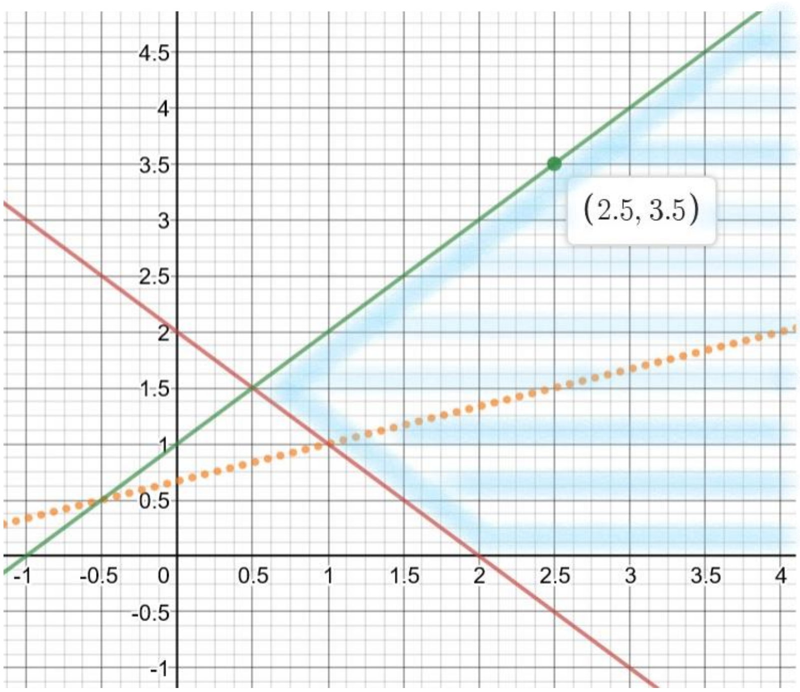
$X_1 + X_2 \geq 2$

$X_2 - X_1 \leq 1$

$X_1 \geq 0, X_2 \geq 0$

Answer:

$X_1 + X_2 = 2$ >> (0,2) and (2,0)
$X_2 - X_1 = 1$ >> (0,1) and (2.5,3.5) also (3,4)



To determine the direction of solution region for each constraints:

Constraint 1:
Let us choose point (2,2)
$X_1 + X_2 \geq 2$
$4 \geq 2$
Constraint 2:
Let us choose point (3,2)
$X_2 - X_1 \leq 1$
$-1 \leq 1$
Let us choose point (1,2.5)
$1.5 \nleq 1$
To plot objective function line, which pass through pint (1,1)
$-2X_1 + 6X_2 = 4$
We need other point $X_1 = 0$
$6(X_2) = 4 \gg X_2 = 0.667$
(0,0.667) , (4,2), (1,1)

https://www.desmos.com/calculator/cx3mcyyye3

The solution space is unbounded in direction of X_1, and the value of Z can be increased indefinitely (Unbounded Solution).

2- Max $Z = -X_1 + X_2$

Subject to

(1) $-X_1 + 4X_2 \geq 0$

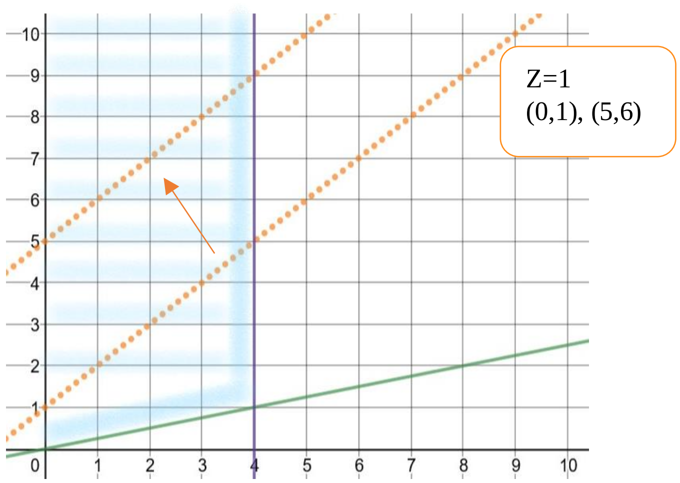
(2) $X_1 \leq 4$

(3) $X_1 \geq 0, X_2 \geq 0$

Answer:

$-X_1 + 4X_2 = 0$ >> (0,0) and (4,1) also (8,2), (12,3)

$X_1 = 4$



To determine the direction of solution region for each constraints:

Constraint 1:
Let us choose point (3,2)

$-X_1 + 4X_2 \geq 0$

$5 \geq 0$

To plot objective function line, which pass through pint (0,1)

$-X_1 + X_2 = 1$

We need other point $X_1 = 5$

$X_2 = 6$

The solution space is unbounded in direction of X_2, and the value of Z can be increased indefinitely.

HW

3- Max $Z = 2X_1 + X_2$

Subject to:

$X_1 - X_2 \leq 10$

$2X_1 \leq 40$

$X_1 \geq 0, X_2 \geq 0$

2- Infeasible (No Solution)

1-Max $Z = 200X_1 + 300X_2$

Subject to

$0.2X_1 + 0.3X_2 \geq 15$

$0.1X_1 + 0.1X_2 \leq 4$

$0.5X_1 + 0.15X_2 \geq 9$

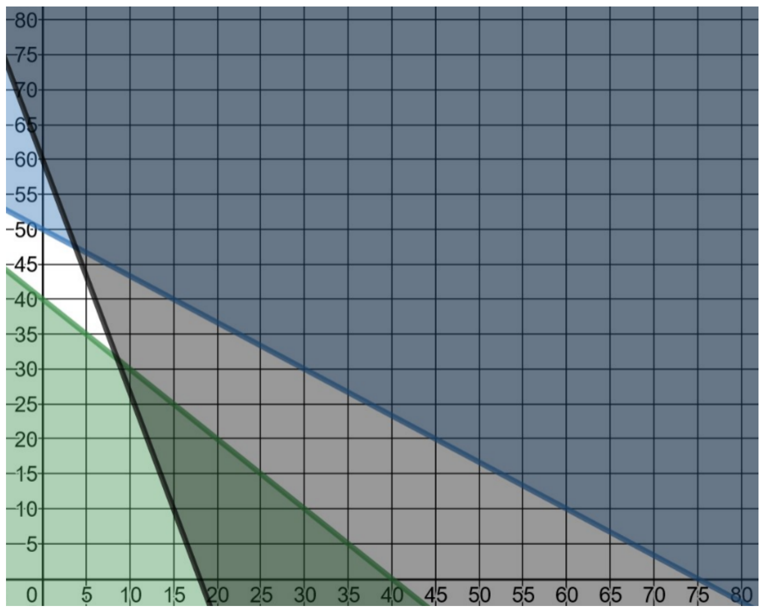
$X_1 \geq 0, X_2 \geq 0$

To determine tow point on each straight line

$0.2X_1 + 0.3X_2 = 15$ >>
(0,50) and (75,0)

$0.1X_1 + 0.1X_2 = 4$ >>
(0,40) and (40,0)

$0.5X_1 + 0.15X_2 = 9$ >>
(0,60) and (18,0)



The problem is infeasible.

HW 2-Max $Z = X_1 + X_2$

Subject to

$X_1 - X_2 + 1 \leq 0$

$-X_1 + X_2 + 1 \leq 0$

$X_1 \geq 0, X_2 \geq 0$

3-Multiple Optimal solution

1-Max $Z = 200X_1 + 400X_2$

Subject to

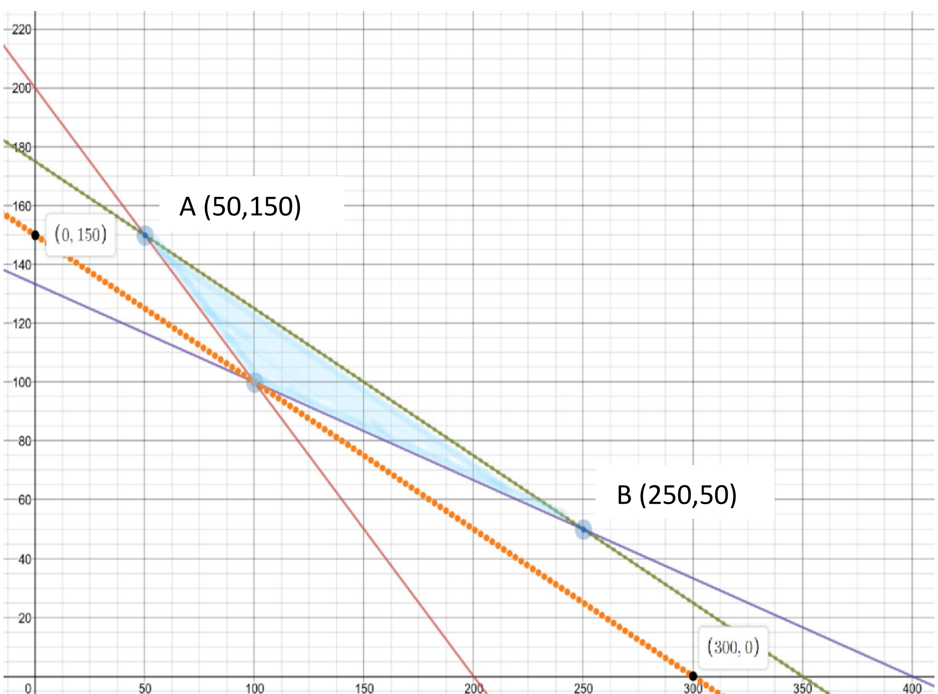
$(1)\, X_1 + X_2 \geq 200$
$(2)\, X_1 + 3X_2 \geq 400$
$(3)\, X_1 + 2X_2 \leq 350$
$(4)\, X_1 \geq 0, X_2 \geq 0$

To determine tow point on each straight line

$X_1 + X_2 = 200$ >> (0,200) and (200,0)

$X_1 + 3X_2 = 400$ >> (0,133.3) and (400,0)

$X_1 + 2X_2 = 350$ >> (0,175) and (350,0)



Z=60000
(0,150), (300,0)

Z=70000
(350,0), (0,175)

To determine the direction of solution region for each constraint:
Constraint 1:
Let us choose point (120,150)
$X_1 + X_2 \geq 200$
$270 \geq 200$
Constraint 2:
Let us choose point (150,100)
$X_1 + 3X_2 \geq 400$
$450 \geq 400$
Constraint 3:
Let us choose point (150,120)
$X_1 + 2X_2 \leq 350$
$390 \nleq 350$
To plot objective function line, which pass through pint (100,100)
$200X_1 + 400X_2 = 60000$
We need other point $X_2 = 0$
$200(X_1) = 60000$
$X_1 = 300$
(300,0), (0,150)

https://www.desmos.com/calculator/bbwpydvaxv

The problem has (Multiple Optimal solution) infinite number of optimal solutions. Any point on the line segment (AB) represents an alternative optimum with the same objective value z= 70000.

(-)* $X_1 + X_2 = 200$

$\underline{X_1 + 2X_2 = 350}$

$X_2 = 350 - 200 = 150$

$X_1 = 50$

$X_1 + 3X_2 = 400$

$\underline{(-) * \ X_1 + 2X_2 = 350}$

$X_2 = 400 - 350 = 50$

$X_1 = 250$

(-)* $X_1 + X_2 = 200$

$\underline{X_1 + 3X_2 = 400}$

$2X_2 = 400 - 200$

$X_2 = 100, X_1 = 100$

(X_1, X_2)	Z
(100,100)	60000
(50,150)	70000
(250,50)	70000

The problem has Multiple Optimal solution.

HW

2-Min $Z = 3X_1 + 2X_2$

Subject to

$-X_1 + X_2 \leq 2$

$3X_1 + 2X_2 \geq 12$

$X_1 \geq 0, X_2 \geq 0$

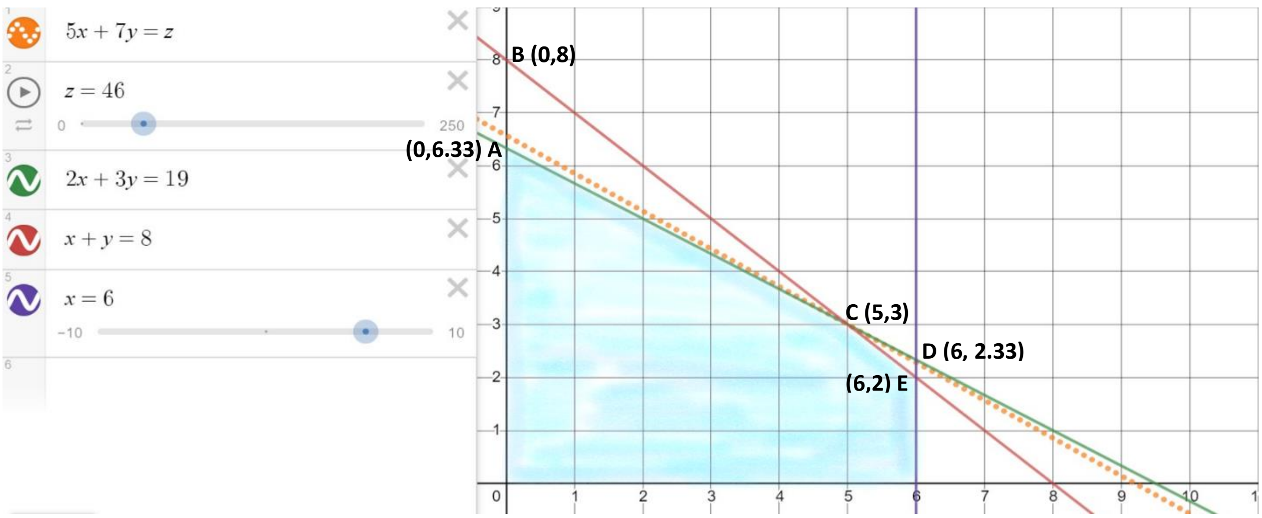
Example: suppose we have LPP

$$\boldsymbol{Max} \quad \boldsymbol{Z = 5x_1 + 7x_2}$$

Constraints:

$$\boldsymbol{x_1 \le 6}$$
$$\boldsymbol{2x_1 + 3x_2 \le 19}$$
$$\boldsymbol{x_1 + x_2 \le 8}$$
$$\boldsymbol{x_1, x_2 \ge 0}$$

1- Determine the optimum solution by use the graphical method.



https://www.desmos.com/calculator/zrwojljtxm

The optimal solution at point (C) $x_1^* = 5$, $x_2^* = 3$ with an optimal value Z=46 $

2- Discuss the sensitivity analysis for the constraints?

First classify the constraints:

$x_1^* = 5$, $x_2^* = 3$, Z(C) = 46

$x_1^* = 5 \le 6$ is non-binding constraint (available resource)

$2x_1^* + 3x_2^* = 19$ is binding constraint (rare resource)

$x_1^* + x_2^* = 8$ is binding constraint (rare resource)

First Constraint $x_1 \le 6$:

Should move parallel to itself until it passes through the optimum solution (c). The new form of constraint is $\boldsymbol{x_1 \le 5}$

The minimum reduced value = 6-5= 1 unit

Second Constraint $2x_1 + 3x_2 \le 19$:

Should move parallel to itself until it passes through the point B (0,8).

$$\mathbf{2(0) + 3(8) = 24}$$

The new form of constraint is $\mathbf{2x_1 + 3x_2 \le 24}$

The new optimum solution is B (0,8).

The new maximum of objective Z(B)= $\mathbf{5(0) + 7(8) = 56}$ $

The maximum increasing= $\Delta_2 = 24 - 19 = 5$ unit

The shadow price = $\frac{56-46}{5} = 2$ $

Third Constraint $x_1 + x_2 \le 8$:

Should move parallel to itself until it passes through the point D (6,2.33).

$$\mathbf{6 + 2.33 = 8.33}$$

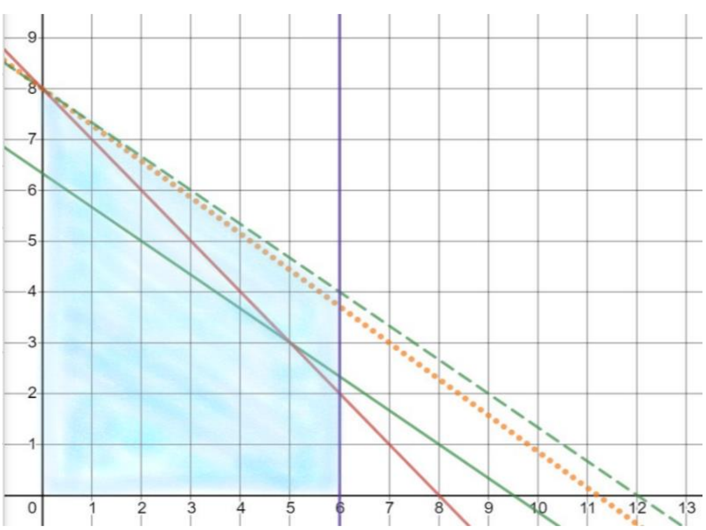
The new form of constraint is $\mathbf{x_1 + x_2 \le 8.33}$

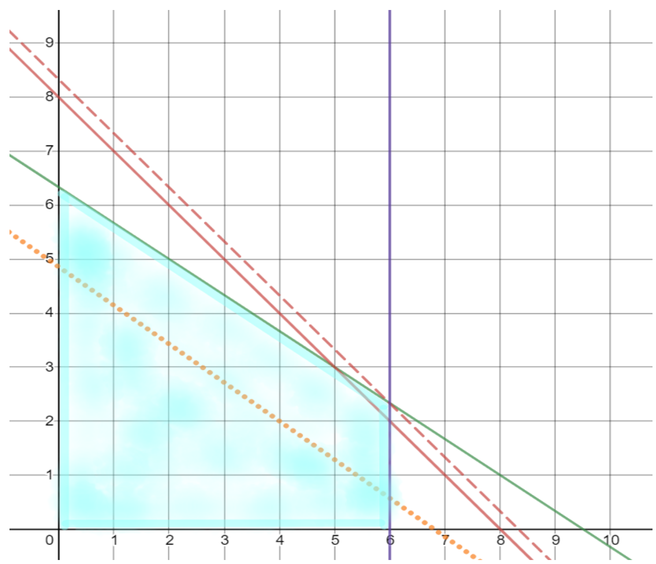
The new optimum solution is D (6, 2.33).

The new maximum of objective Z(B)= $\mathbf{5(6) + 7(2.33) = 46.31\ \$}$

The maximum increasing= $\Delta_2 = 8.33 - 8 = 0.33$ unit

The shadow price = $\frac{46.31-46}{0.33} = \frac{31}{33} = 0.94$ $





3- Determine the optimality condition that will keep the optimum unchanged. Also, Determine the optimality rang for C1 and C2, assuming that the other coefficient is kept constant at its present value.?! (sensitivity analysis for the cofficient of the objective function)

The slop of $z = c_1x_1 + c_2x_2$ is $-\frac{c_1}{c2}$

The optimum solution will remain at point C so long as $z = c_1x_1 + c_2x_2$ lies between the two lines $2x_1 + 3x_2 \leq 19$ and $x_1 + x_2 \leq 8$

The slope of the binding constraint $\boldsymbol{2x_1 + 3x_2 \leq 19}$ is $-\frac{2}{3}$

The slope of the binding constraint $\boldsymbol{x_1 + x_2 \leq 8}$ is -1

$$-1 \leq \frac{-C_1}{C_2} \leq \frac{-2}{3} \qquad \gg \frac{2}{3} \leq \frac{C_1}{C_2} \leq 1$$

Range of Optimality for *C1* (with *C2* staying 7)

$$\frac{2}{3} \leq \frac{C_1}{7} \leq 1$$
$$4.67 \leq C_1 \leq 7$$

Range of Optimality for *C2* (with *C1* staying 5)

$$\frac{2}{3} \leq \frac{5}{C_2} \leq 1$$
$$1 \leq \frac{C_2}{5} \leq \frac{3}{2}$$
$$5 \leq C_2 \leq 7.5$$

4- Suppose that the unit revenues c1 and c2 are changed to 6$ and 7.5$, respectively. Will the current optimum remain the same?

The new objective function is $Max \quad Z = 6\,x_1 + 7.5\,x_2$

The solution at (C) will remain optimal because $\frac{6}{7.5} = 0.8\ \epsilon\ [0.67,1\ ]$.

Notice that although the values of the variables at optimum point C remain unchanged, the optimum value of Z changes to $6(5) + 7.5(3) = 52.5$ $.

HW

Example: suppose we have LPP

Objective function:

Maximize $Z = x_1 + 1.5x_2$

Constrains:

(1) $2x_1 + 2x_2 \leq 16$

(2) $x_1 + 2x_2 \leq 12$

(3) $4x_1 + 2x_2 \leq 28$

(4) $x_1 , x_2 \geq 0$

Discuss the sensitivity analysis?