Chapter1: Basic concepts of Probability and random variable

Addition rule: $\boldsymbol{P(A)+P(B)-P(A\cap B)}$

Disjoint Events :

$If\ A\cap B\ =\phi\ ;\ we\ say\ that\ A\ and\ B\ are\ \boldsymbol{disjoint}\ sets\ and\ P(A\cap B)=0$

Complementary events: $P(A)=1-P(A^C)$

Conditional probability:

$P(A|B)=\frac{P(A\cap B)}{P(B)}\quad ;\quad P(B|A)=\frac{P(A\cap B)}{P(A)}$

$P(A^c|B)=1-P(A|B)$

Independent Events: $\begin{cases} P(A\cap B)=P(A)P(B) \\ P(A|B)=P(A) \\ P(B|A)=P(B) \end{cases}$

De Morgan's laws

$P(A\cap B)^C=P(A^C\cup B^C)$

$P(A\cup B)^C=P(A^C\cap B^C)$

$P(A^C\cap B^C)=P(A\cup B)^C\ =1-P(A\cup B)$

Another rule:

$$P(A\cup B)=\begin{cases} P(A)+P(B)-P(A\cap B) \\ P(A)+P(A^C\cap B) \\ P(A^C\cap B^c)^c=1-P(A^C\cap B^C) \end{cases}$$

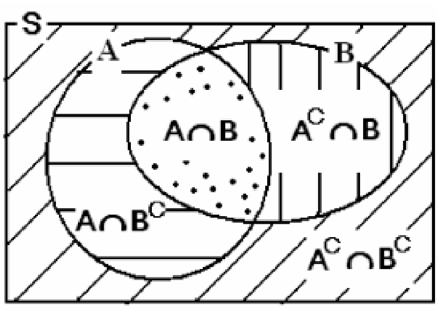
$$P(A\cap B)=P(A^C\cup B^C)^C=1-P(A^C\cup B^C)=1-[P(A^C)+P(B^C)-P(A^C\cap B^C)]$$

$P(A)=P(A\cap B)+P(A\cap B^C)$

$P(B)=P(A\cap B)+P(A^C\cap B)$

$P(A\cap B^C)=P(A)-P(A\cap B)$

$P(A^c\cap B)=P(B)-\ P(A\cap B)$



Q1: Consider the experiment of flipping a balanced coin **three** times independently.

1. The number of points in the sample space is..
   (A) 2 (B) 6 (C) 8 (D) 3 (E) 9
2. The probability of getting exactly two heads is…
   (A) 0.125 (B) 0.375 (C) 0.667 (D) 0.333 (E) 0.451
3. The events 'exactly two heads' **and** 'exactly three heads' are…
   (A) Independent (B) disjoint (C) equally likely (D) identical (E) None
4. The events 'the first coin is head' **and** 'the second and the third coins are tails' are…
   (A) Independent (B) disjoint (C) equally likely (D) identical (E) None

Solution :

$S = \{H,T\} \times \{H,T\} \times \{H,T\}$

$s = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

1) $n(s) = 2 \times 2 \times 2 = 2^3 = 8$

2) *Event A: getting exactly two heads*

$A = \{HHT, HTH, THH\} \to P(A) = \frac{n(A)}{n(S)} = \frac{3}{8} = 0.375$

3) *Event B: exactly three heads* , $B = \{HHH\}$

$\therefore A \text{ and } B = A \cap B = \phi \ \to \therefore A \text{ and } B \text{ are disjoint}$

4) *Event C: the first coin is head* , $C = \{HHH, HHT, HTH, HTT\} \to P(C) = \frac{4}{8} = \frac{1}{2}$

Event D: the second and the third coins are tails, $D = \{HTT, TTT\} \to P(D) = \frac{2}{8} = \frac{1}{4}$

$C \text{ and } D = C \cap D = \{HTT\} \to P(C \cap D) = \frac{1}{8}$

$\text{as } P(C \cap D) \neq 0 \ \therefore C \text{ and } D \text{ are } \textbf{not} \text{ disjoint}$

$\text{as } P(C) \neq P(D) \therefore C \text{ and } D \text{ are } \textbf{not} \text{ equaly likely}$

$\text{as } P(C \cap D) = P(C)P(D) = \frac{1}{8} \ \therefore C \text{ and } D \text{ are independent.}$

Q2. Assume that $P(A) = 0.3$, $P(B) = 0.4$, $P(A \cap B \cap C) = 0.03$, and $P(\overline{A \cap B}) = 0.88$, then

1. the events A and B are,
   (A) Independent (B) Dependent (C) Disjoint (D) None of these.
2. $P(C|A \cap B)$ is equal to,
   (A) 0.65 (B) 0.25 (C) 0. 35 (D) 0.14

Solution:

1) $P(A \cap B) = 1 - P(A \cap B)^C = 1 - 0.88 = 0.12 \neq 0$

$\therefore A \text{ and } B \text{ are not disjoint}$

$P(A).P(B) = 0.3 \times 0.4 = 0.12 = P(A \cap B)$

$\therefore A \text{ and } B \text{ are independent}$

2) $P(C|A \cap B) = \frac{P(C \cap (A \cap B))}{P(A \cap B)} = \frac{P(C \cap A \cap B)}{P(A \cap B)} = \frac{0.03}{0.12} = 0.25$

Q3. If the probability that it will rain tomorrow is 0.23, then the probability that it will not rain tomorrow is:

(A) –0.23 (B) 0.77 (C) –0.77 (D) 0.23

Solution:

A: it will rain tomorrow , $P(A) = 0.23 \rightarrow P(A^C) = P(\bar{A}) = 1 - P(A) = 1 - 0.23 = 0.77$

Q4. The probability that a factory will open a branch in Riyadh is 0.7, the probability that it will open a branch in Jeddah is 0.4, and the probability that it will open a branch in either Riyadh or Jeddah or both is 0.8. Then, the probability that it will open a branch:

1. in both cities is:
   (A) 0.1 (B) 0.9 (C) 0.3 (D) 0.8
2. in neither city is:
   (A) 0.4 (B) 0.7 (C) 0.3 (D) 0.2

Solution:

A: factory open a branch in Riyadh $\rightarrow P(A) = 0.7$

B: factory open a branch in Jeddah $\rightarrow P(B) = 0.4$

$P(A \cup B) = 0.8$

1) $P(A \cap B) = ??$
   Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

   $0.8 = 0.7 + .04 - P(A \cap B)$

$P(A \cap B) = 0.7 + 0.4 - 0.8 = 0.3$

2) the probability that it will open a branch neither in Riyadh nor Jeddah
   $P(A^c \cap B^c) = P(A \cup B)^C = 1 - P(A \cup B) = 1 - 0.8 = 0.2$

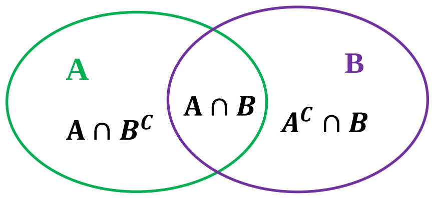
<u>Or by table</u>

	A	A^C	SUM
B	$P(A \cap B) = 0.3$	$P(A^C \cap B) = 0.1$	$P(B) = 0.4$
B^C	$P(A \cap B^C) = 0.4$	$P(A^C \cap B^C) = 0.2$	$P(B^C) = 0.6$
SUM	$P(A) = 0.7$	$P(A^C) = 0.3$	1

Q5. Two engines operate independently, if the probability that an engine will start is 0.4, and the probability that the other engine will start is 0.6, then the probability that both will start is:

(A) 1 (B) 0.24 (C) 0.2 (D) 0.5

Solution :



A: the first engine start → P(A)=0.4

B: the second engine start → P(B)=0.6

the two engines operate independent $\therefore P(A \cap B) = P(A).P(B) = 0.4 * 0.6 = 0.24$

Q6. If P(B) = 0.3 and P(A| B) = 0.4 , then P(A∩ B) equals to;

(A) 0.67 (B) 0.12 (C) 0.75 (D) 0.3

Solution :

$P(A \cap B) = P(A|B)P(B)$; $\left[from\ P(A|B) = \dfrac{P(A \cap B)}{P(B)} \right]$

$P(A \cap B) = 0.4 * 0.3 = 0.12$

Q7. From a box containing 4 black balls and 2 green balls, 3 balls are drawn independently in succession, each ball being replaced in the box before the next draw is made.

The probability of drawing 2 green balls and 1 black ball is:

(A) 6/27 (B) 2/27 (C) 12/27 (D) 4/27

Solution :

3 balls (independently and with replaced)

g: green ball ; $P(g) = \frac{2}{6} = \frac{1}{3}$

b: black ball ; $P(b) = \frac{4}{6} = \frac{2}{3}$

$$S = \{bbb, bbg, bgb, gbb, ggb, gbg, bgg\}$$

$P(2g\ and\ 1b) = P(\{bgg, gbg, ggb\}) = P(b \cap g \cap g) + P(g \cap b \cap g) + P(g \cap g \cap b)$

Since the balls are drawn **independently**

$$= P(b)P(g)P(g) + P(g)P(b)P(g) + P(g)P(g)P(b)$$

$$= 3\left[\left(\frac{4}{6}\right)\left(\frac{2}{6}\right)\left(\frac{2}{6}\right)\right] = 3\left[\frac{2}{27}\right] = \frac{6}{27}$$

Q8. If $P(A_1) = 0.4$, $P(A_1 \cap A_2) = 0.2$, and $P(A_3|A_1 \cap A_2) = 0.75$, then

1. $P(A_2|A_1)$ equals to
   (A) 0.00 (B) 0.20 (C) 0.08 (D) 0.50
2. $P(A_1 \cap A_2 \cap A_3)$ equals to
   (A) 0.06 (B) 0.35 (C) 0.15 (D) 0.08

Solution :

1) $\mathrm{P}(\mathrm{A}_2|\mathrm{A}_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{0.2}{0.4} = 0.5$

2) $P(A_3|A_1 \cap A_2) = \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)} \rightarrow P(A_1 \cap A_2 \cap A_3) = P(A_3|A_1 \cap A_2)P(A_1 \cap A_2)$

$$= 0.75 * 0.2 = 0.15$$

Q9. If P(A)=0.9, P(B)=0.6, and P(A$\cap$B)=0.5, then:

1. $P(A \cap B^C)$ equals to
   (A) 0.4 (B) 0.1 (C) 0.5 (D) 0.3
2. $P(A^C \cap B^C)$ equals to
   (A) 0.2 (B) 0.6 (C) 0.0 (D) 0.5
3. P(B|A) equals to
   (A) 0.5556 (B) 0.8333 (C) 0.6000 (D) 0.0
4. The events A and B are
   (A) independent (B) disjoint (C) joint (D) none
5. The events A and B are
   (A) disjoint (B) dependent (C) independent (D) none

Solution :

1) $P(A \cap B^C) = P(A) - P(A \cap B) \quad [from\ P(A) = P(A \cap B) + P(A \cap B^C)]$

2) $P(A^C \cap B^C) = P(A \cup B)^C = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$

$$= 1 - [0.9 + 0.6 - 0.5] = 1 - 1 = 0$$

3) $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.5}{0.9} = 0.556$

4) $P(A \cap B) = 0.5$

5) $P(A \cap B) \neq P(A)P(B) \rightarrow 0.5 \neq 0.9 * 0.6 \ \therefore A\ and\ B\ are\ dependent\ .$

$P(A \cap B) \neq 0 \ \therefore A\ and\ B\ are\ joint\ .$

Q10: A pair of fair dice is rolled. Let X denote the sum of the number of dots on the top faces.

Construct the probability distribution of X for a paid of fair dice.

Find

(1) $P(X \geq 11)$
(2) $P(X<4)$
(3) $P(X=8)$
(4) Find the probability that X takes an even value.

Solution :

The sample space of equally likely outcomes where the first digit is die 1 and the second number is die 2

$\boldsymbol{S}$ = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)

(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)

(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)

(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

The possible values of X = 2,3,4,…,12

This table is the probability distribution of X

X	2	3	4	5	6	7	8	9	10	11	12
Number of ways	1	2	3	4	5	6	5	4	3	2	1
P(X)	$\mathbf{\frac{1}{36}}$	$\mathbf{\frac{2}{36}}$	$\mathbf{\frac{3}{36}}$	$\mathbf{\frac{4}{36}}$	$\mathbf{\frac{5}{36}}$	$\mathbf{\frac{6}{36}}$	$\mathbf{\frac{5}{36}}$	$\mathbf{\frac{4}{36}}$	$\mathbf{\frac{3}{36}}$	$\mathbf{\frac{2}{36}}$	$\mathbf{\frac{1}{36}}$

(1) P(X=8)=p((4,4),(3,5),(5,3),(2,6),(6,2))= $\frac{5}{36}$
(2) $P(X\geq11)$= P(X=11)+ P(X=12)= $\frac{2}{36}+\frac{1}{36}=\frac{3}{36}$
(3) P(X<5)= P(X=2)+ P(X=3)+ P(X=5)=)= $\frac{1}{36}+\frac{2}{36}+\frac{3}{36}=\frac{5}{36}$
(4) Find the probability that X takes an even value.

=P(X=2)+P(X=4)+P(X=6)+P(X=8)+P(X=10)+P(X=12)

$=\frac{1}{36}+\frac{3}{36}+\frac{5}{36}+\frac{5}{36}+\frac{3}{36}+\frac{1}{36}=\frac{18}{36}=0.5$

Q11: Let f(x) be probability distribution of random variable X

x	-2	-1	0	1	2
f(x)	0.1	c	0.3	0.2	0.2

(1) Find the value of c
(2) Find F(x)

Solution :

(1) Find the value of c

Since f(x) is probability distribution the $\sum f(x) = 1$

0.1+c+0.3+0.2+0.2=1

C=0.2

(2) Find F(x)

x	-2	-1	0	1	3
f(x)	0.1	0.2	0.3	0.2	0.2
F(x)=$P(X \leq x)$	0.1	0.3	0.6	0.8	1

Q12:Let M>0, and suppose $f(x) = cx^2 \; for \; 0 < x < M, otherwise \; f(x) = 0$

(1) What is the value of c if f is density? (depending on M)

Since f is density, then $\int_0^M cx^2dx = 1$

$$c\left[\frac{x^3}{3}\right]_0^M = 1$$

$$c\frac{(M^3-0^3)}{3}=1$$

$$c = \frac{3}{M^3}$$

(2) Find the cumulative distribution

$$\int_0^x f(t)dt = \int_0^x ct^2dt = \int_0^x \frac{3}{M^3}t^2dt = \frac{3}{M^3}\left[\frac{t^3}{3}\right]_0^x$$

$$= \frac{3}{M^3}\left(\frac{x^3}{3} - \frac{0^3}{3}\right)$$

$$= \frac{x^3}{M^3} \quad , \; for \; all \; 0 < x < M \; .$$

Q13. A total of 36 members of a club play tennis, 28 play squash, 18 play badminton, 22 play both tennis and squash, 12 play both tennis and badminton, 9 play both squash and badminton, 4 play all 3. What is the probability that at least one a member of this club plays at least one sport. Assuming that the total number of members in the club is 50.

Solution:
T:people who play tennis ; n(T)=36
S:people who play squash ; n(S)=28
B:people who play badminton ; n(B)=18

$n(T \cap S) = 22 \;\; ; \;\; n(T \cap B) = 12 \;\; ; \;\; n(S \cap B) = 9 \;\; ; \;\; n(T \cap S \cap B) = 4$
number of members in the club= N=50

$$P(T \cup S \cup B) = P(T) + P(S) + P(B) - P(T \cap S) - P(T \cap B) - P(S \cap B) + P(T \cap S \cap B)$$
$$= \frac{36 + 28 + 18 - 22 - 12 - 9 + 4}{50} = \frac{43}{50}$$

NOTE:

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = \sum_{j=1}^{n} P(E_j) - \sum_{j_1<j_2} P(E_{j1} \cap E_{j2}) + \cdots$$
$$+(-1)^{r+1} \sum_{j_1<j_2<\cdots<jr} P(E_{j1} \cap E_{j2} \cap \ldots \cap E_{jr}) + \cdots + (-1)^{n+1} P(E_1 \cap E_2 \cap \ldots \cap E_n)$$

The summation $\sum_{j_1<j_2<\cdots<jr} P(E_{j1} \cap E_{j2} \cap \ldots \cap E_{jr})$ is taken over all of the $\binom{n}{r}$ possible subset of size (r) of the set {1,2,…,n}

Q14. Suppose we have a fuse box containing 20 fuses of which 5 are defective D and 15 are non-defective N. If 2 fuses are selected at random and removed from the box in succession without replacing the first.

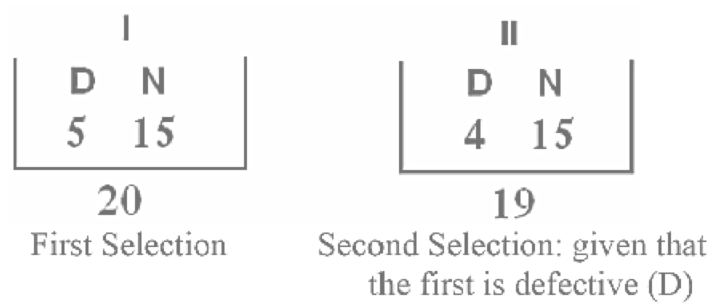
what is the probability that both fuses are defective.

Solution:
A={ the first fuse is defective } ; B={ the second fuse is defective }

The probability of first removing a defective fuse is P(A)=5/20

The probability of removing a second defective fuse after a defective first fuse was removed is $P(B|A) = \frac{4}{19}$
A∩B={ the event that **A** occurs and then **B** occurs after **A** occurred}
={both fuses are defective}

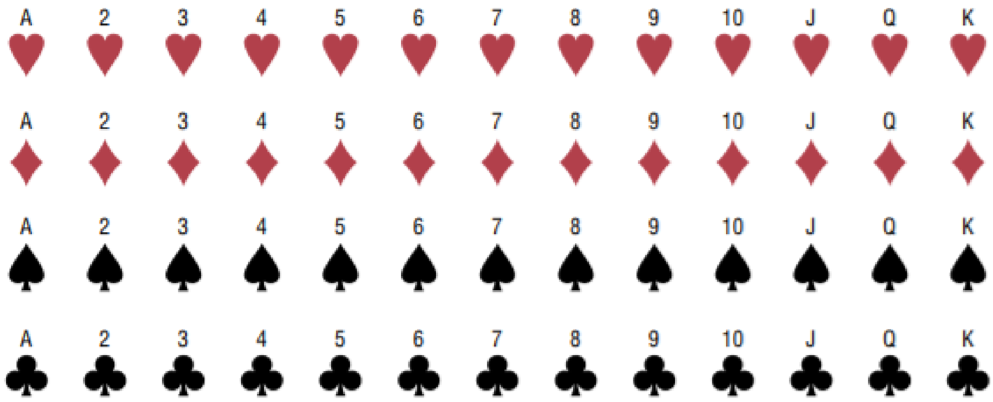
$$P(A \cap B) = P(A)P(B|A) = \frac{5}{20}\frac{4}{19} = 0.052632$$



Q15. Three cards are drawn in succession, **without replacement**, from an ordinary deck of playing cards. Fined $P(A_1 \cap A_2 \cap A_3)$, where the events A1, A2 , and A3 are defined as follows:

A1 = {the 1-st card is a red ace}
A2 = {the 2-nd card is a 10 or a jack}
A3 = {the 3-rd card is a number greater than 3 but less than 7}

Solution:

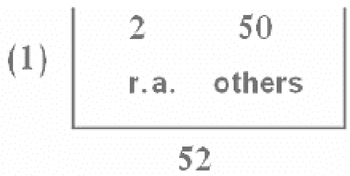


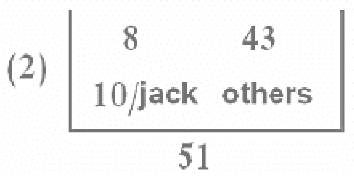
Playing card (52) $\begin{cases} \text{13 cards of hearts ♥} \\ \text{13 cards of diamonds ♦} \\ \text{13 cards of spades ♠} \\ \text{13 cards of clubs ♣} \end{cases}$; every set has A,J,Q,K,2,3,4,5,6,7,8,9,10 .

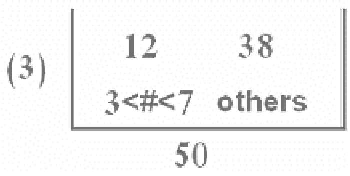
Where A: Ace
J: Jack
Q:Queen
K:king

Then,
A1 = {the 1-st card is a red ace} $n(A_1) = 2$
A2 = {the 2-nd card is a 10 or a jack} $n(A_2) = 8$
A3 = {the 3-rd card is a number greater than 3 but less than 7} $n(A_3) = 12$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_3) = \frac{2}{52}\ \frac{8}{51}\ \frac{12}{50} = 0.0014479$$







Q16. **(HW)** Suppose that a fair die is thrown twice independently, then

1. the probability that the sum of numbers of the two dice is less than or equal to 4 is;
   (A) 0.1667 (B) 0.6667 (C) 0.8333 (D) 0.1389
2. the probability that at least one of the die shows 4 is;
   (A) 0.6667 (B) 0.3056 (C) 0.8333 (D) 0.1389
3. the probability that one die shows one and the sum of the two dice is four is;
   (A) 0.0556 (B) 0.6667 (C) 0.3056 (D) 0.1389
4. the event A={the sum of two dice is 4} and the event B={exactly one die shows two} are,
   (A) Independent (B) Dependent (C) Joint (D) None of these.

Solution of Q2:

$$S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$$
$$= \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$$
$$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$$
$$(3,1),\dots,(4,1),\dots,(5,1),\dots,$$
$$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

1) A: *the sum of numbers of the two dice* ≤ 4

$$A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\} \rightarrow P(A) = \frac{6}{36} = \frac{1}{6} = 0.1667$$

2) B: *at least one of the die shows* 4

$$B = \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4),(4,1),(4,2),(4,3),(4,5),(4,6)\}$$

$P(B) = \frac{11}{36} = 0.3056$

3) C: *that one die shows* (1) *and the sum of the two dice* $= 4$

$$C = \{(1,3),(3,1)\} \rightarrow P(C) = \frac{2}{36} = 0.0556$$

4) D: *the sum of two dice* $= 4$, $D = \{(1,3),(3,1),(2,2)\} \rightarrow P(D) = \frac{3}{36} = 0.0833$

E: *one die shows* 2

$E = \{(1,2),(2,1),(2,3),(2,4),(2,5),(2,6),(3,2),(4,2),(5,2),(6,2)\}$

$P(E) = \frac{10}{36} = 0.2778$

$D \cap E = \{\ \} \rightarrow P(D \cap E) = 0 \;\therefore D \text{ and } E \text{ are disjoint.}$

$P(D \cap E) = 0 \neq P(D)P(E) \rightarrow D \text{ and } E \text{ are dependent.}$

Q17. **(HW)** The probability that a computer system has an electrical failure is 0.15, and the probability that it has a virus is 0.25, and the probability that it has both problems is 0.10, then the probability that the computer system has the electrical failure or the virus is:

(A) 1.15 (B) 0.2 (C) 0.15 (D) 0.30

Solution :

Event A: The computer system has an electrical failure
Event B:The computer has a virus

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.15 + 0.25 - 0.1 = 0.3$$

Q18. **(HW)** Suppose that the experiment is to randomly select with replacement 2 children and register their gender (B=boy, G=girl) from a family having 2 boys and 6 girls.

1. The number of outcomes (elements of the sample space) of this experiment equals to
   (A) 4 (B) 6 (C) 5 (D) 125
2. The event that represents registering at most one boy is
   (A){GG, GB, BG} (B) {GB, BG} (C){GB}C (D) {GB, BG, BB}
3. The probability of registering no girls equals to
   (A) 0.2500 (B) 0.0625 (C) 0.4219 (D) 0.1780
4. The probability of registering exactly one boy equals to
   (A) 0.1406 (B) 0.3750 (C) 0.0141 (D) 0.0423
5. The probability of registering at most one boy equals to

   6 gairls
   2 boys

   2 Children with replacement (the events are considered to be independent)

   (A) 0.0156 (B) 0.5000 (C) 0.4219 (D) 0.9375

Solution :

G:$Girl$ and **B**: Boy

$$n = 8\ ;\ P(G) = \frac{6}{8} = 0.75\ ;\ P(B) = \frac{2}{8} = 1 - 0.75 = 0.25$$

1) S={G,B}×{G,B}={GG,GB,BG,BB} ; n(S)=2×2=4

2) A= at most one boy={ GG,GB,BG}

5) P(A)=P({ GG , GB , BG})

$and \cong \cap \cong \times$ $or \cong \cup \cong +$

P(A) =P({GG })+ P({GB })+ P({BG }) [because the independent]

=P(G)P(G)+ P(G)P(B)+ P(G)P(B)

$= (0.75)^2 + (2)(0.75)(0.25) = 0.9375$

3) C= no girls={BB}

P(C)=P({BB})=P(B).P(B)= 0.25(0.25)=0.0625

4) D= exactly one boy= {GB, BG}
P(D)=P({GB, BG})=P({GB})+P({BG}) =2 P(G)P(B)= (2)(0.75)(0.25)=0.375

Q19. **(HW)** If the probability of passing course (A) is 0.6, passing course (B) is 0.7, passing course A or B is 0.9. Find:

1- Probability of passing course A and B.
2- Probability of passing course A only.
3- Probability of passing course B and not passing course A.
4- Probability of not passing course A and B.
5- Probability of passing course B or not passing course A.

Solution: H.W

Q20. The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$.
Find the probability that a plane

1- arrives on time given that it departed on time.
2- departed on time given that it has arrived on time.
3- arrived on time given that it has not departed on time.

Solution : H.W