Math 204
Differential Equations
Exercises 1
First Order Differential Equations

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First Order Differential Equations (Exercises)

1. Initial Value Problem
   (Existence and Uniquenence)
2. Separable Equations
3. Equations with Homogeneous Coefficients
4. Appropriate Substitution
5. Exact Equations

Initial Value Problem
(Existence and Uniquenence)

Existence of a unique solution

Theorem

Consider a first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

with initial condition

$$y(x_0) = y_0$$

Let R *be a rectangular region defined by* $a \leq x \leq b, c \leq y \leq d$ *that contains the point* (x_0, y_0) *in its interior.*
If f *and* $\frac{\partial f}{\partial y}$ *are continuous on* R *then there exists an interval* I *centered at* x_0 *and a unique function* $y(x)$ *satisfies the IVP.*

Initial Value Problem

Example

*[a] Determine the local region in the $xy-$ plane for which the following differential equation has a unique solution through the origin $(0,0)$.

$$\sqrt{9-y^2}\,\frac{dy}{dx} = \ln(4-x^2)$$

[a] Exam Problem

Initial Value Problem

Example

*[a] Determine the local region in the $xy-$ plane for which the following differential equation has a unique solution through the origin $(0,0)$.

$$\sqrt{9-y^2}\,\frac{dy}{dx}=\ln(4-x^2)$$

[a]Exam Problem

Solution:

$$R=\{(x,y):-2<x<2,-3<y<3\}$$

Initial Value Problem

Example

* Find and sketch the local region in the $xy-$ plane for which the following initial value differential equation has a unique solution

$$(x^2 - x - 2)\frac{dy}{dx} = \sqrt{1 - \ln(2 - y)}$$

$$y(0) = 0$$

Initial Value Problem

Example

* Find and sketch the local region in the $xy-$ plane for which the following initial value differential equation has a unique solution

$$(x^2 - x - 2)\frac{dy}{dx} = \sqrt{1 - \ln(2 - y)}$$

$$y(0) = 0$$

Solution:

$$R = \{(x, y) : -1 < x < 2, 2 - e < y < 2\}$$

Initial Value Problem

Example

* Find and sketch the local region in the $xy-$ plane for which the following initial value differential equation has a unique solution

$$\sqrt{x - y^2}\, dx - (2x^2 - 5x + 2)\, dy = 0$$

$$y(1) = 0$$

Initial Value Problem

Example

* Find and sketch the local region in the $xy-$ plane for which the following initial value differential equation has a unique solution

$$\sqrt{x-y^2}\,dx-(2x^2-5x+2)\,dy=0$$

$$y(1)=0$$

Solution:

$$R=\{(x,y):\frac{1}{2}<x<2,y^2<x\}$$

Exam Problems

Determine and sketch the largest local region for which the following initial value problems admits a unique solution

1. $$\begin{cases} \ln(x-2)\,\frac{dy}{dx} = \sqrt{y-2} \\ y\left(\frac{5}{2}\right) = 4 \end{cases}$$

2. $$\begin{cases} (x^2-1)\,dy + \left(3+y+\sqrt{y-4x}\right)dx = 0 \\ y(0) = 2 \end{cases}$$

3. $$\begin{cases} (x-3)\,\frac{dy}{dx} + y\ln x = 2x \\ y(1) = 2 \end{cases}$$

4. $$\begin{cases} (x-2)(x+3)\,\frac{dy}{dx} = 4\ln y \\ y(-5) = 2 \end{cases}$$

Separable Equations

Separable equations

Definition

Consider a first-order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

If we can write it in the form

$$g(x)dx = h(y)dy$$

then it is said to be separable.

To solve separable equations, we integrate each part

$$\int g(x)dx = \int h(y)dy + c$$

Example

* Solve the initial value problem

$$(xy^2 + 4x)dx + (8y - 2x^2y)dy = 0, \qquad |x| < 2$$

$$y(0) = 0$$

Example

* Solve the initial value problem

$$(xy^2 + 4x)dx + (8y - 2x^2y)dy = 0, \qquad |x| < 2$$

$$y(0) = 0$$

Solution:

$$y^2 = 2\sqrt{4 - x^2} - 4$$

Equations with Homogeneous Coefficients

Equations with Homogeneous Coefficients

Definition

A differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is an equation with homogeneous coefficients if both $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

Equations with Homogeneous Coefficients

Definition

A differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is an equation with homogeneous coefficients if both $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

Use the substitution

$$y = ux \qquad \text{or} \qquad x = vy$$

to reduce it to a separable differential equation.

Equations with Homogeneous Coefficients

Example

Solve the initial value problem

$$xy^2\frac{dy}{dx} = y^3 - x^3, \qquad xy \neq 0$$

$$y(1) = 2$$

Example

Solve the initial value problem

$$xy^2\frac{dy}{dx} = y^3 - x^3, \qquad xy \neq 0$$

$$y(1) = 2$$

Solution:

$$\left(\frac{y}{x}\right)^3 + \ln|x^3| = 8$$

Appropriate Substitution

Appropriate Substitution

If we have a differential equation of the form

$$\frac{dy}{dx} = f(ax + by)$$

we use the substitution

$$u = ax + by$$

then

$$\frac{du}{dx} = a + b\frac{dy}{dx}$$

Equations with linear coefficients

Consider the differential equation

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are real constants.

The two lines $a_1x + b_1y + c_1 = 0$, and $a_2x + b_2y + c_2 = 0$ are parallel, or intersected.

Example

* Solve the DE

$$\frac{dy}{dx} = \frac{1 - x - y}{x + y}$$

Appropriate Substitutions

Example

* Solve the DE

$$\frac{dy}{dx} = \frac{1-x-y}{x+y}$$

Solution:

$$(x+y)^2 = 2x + c$$

Exact Equations

Exact Equations

Definition

A differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is called exact, if there is a function $f(x, y)$ such that

$$df(x, y) = M(x, y)dx + N(x, y)dy = 0$$

Exact Equations

Theorem

If $M, N, \frac{\partial M}{\partial y}$, *and* $\frac{\partial N}{\partial x}$ *are continuous on a region* R *in* $xy-$*plane, then the differential equation*

$$M(x,y)dx + N(x,y)dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example

Solve the problem

$$(e^x + y)dx + (2 + x + ye^y)dy = 0$$

Exact Equations

Example

Solve the problem

$$(e^x + y)dx + (2 + x + ye^y)dy = 0$$

Solution:

$$2y + xy + ye^y - e^y + e^x = c$$

Integrating factor

In order to find the integrating factor, we have tow cases

1. $\mu = \mu(x)$, then
$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$
2. $\mu = \mu(y)$, then
$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

Example

* Verify that the differential equation

$$\cos x\,dx + \left(2 + \frac{3}{y}\right)\sin x\,dy = 0$$

$$0 < x < \pi, \quad y > 0$$

is not exact. Find a suitable integrating factor to convert it to an exact equation, and solve it.

Integrating factor

Example

* Verify that the differential equation

$$\cos x\, dx + \left(2 + \frac{3}{y}\right) \sin x\, dy = 0$$

$$0 < x < \pi, \quad y > 0$$

is not exact. Find a suitable integrating factor to convert it to an exact equation, and solve it.

Solution:

$$2y + 3\ln y + \ln \sin x = c$$

Example

* Solve

$$(y - 2\sqrt{xy})dx + xdy = 0, \qquad x > 0, y > 0$$

Example

* Solve

$$(y - 2\sqrt{xy})dx + xdy = 0, \qquad x > 0, y > 0$$

$$\sqrt{xy} = x + c$$

Example

* By using an appropriate substitution find the general solution of

$$\frac{dy}{dx} = \frac{y}{x}\ln(xy), \qquad x > 0, y > 0$$

Example

* By using an appropriate substitution find the general solution of

$$\frac{dy}{dx} = \frac{y}{x} \ln(xy), \qquad x > 0, y > 0$$

Solution:

$$u = xy$$

$$\ln|\ln(xy) + 1| = \ln x + c$$

Example

Solve
$$\frac{dy}{dx} = (4x + y + 5)^2$$

Example

Solve

$$\frac{dy}{dx} = (4x + y + 5)^2$$

Solution:

$$\frac{1}{2}\tan^{-1}\left(\frac{4x + y + 5}{2}\right) = x + c$$

Example

Solve

$$(2xy + 2xy \ln y)\, dx + (2 + \ln y)(5 - x^2)\, dy = 0$$

Example

Solve

$$(2xy + 2xy \ln y)\, dx + (2 + \ln y)(5 - x^2)\, dy = 0$$

$$y(1 + \ln y) = c|5 - x^2|$$

Example

* Solve

$$\frac{dy}{dx} = 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}, \qquad x > 0,\ y > 0$$

$$y(1) = 1$$

Example

* Solve

$$\frac{dy}{dx} = 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}, \qquad x > 0,\, y > 0$$

$$y(1) = 1$$

$$y - \ln(1 + y) = x + \ln x - \ln 2$$

Example

Solve

$$(y - xy)dy - (x + y^2)dx = 0, \qquad x > 1$$

Example

Solve

$$\tan y - x\frac{dy}{dx} - 4x^2\tan y = 0, \qquad x > 0, y \in (0, \pi)$$

Example

Solve the problem

$$(6x + y^2)dx + y(2x - 3y)dy = 0$$

Example

Solve

$$y' = 3 - \sqrt{x + y - 1}$$

$$y(0) = 1$$

Example

Solve

$$\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$$

Example

Solve

$$\frac{dy}{dx} = xy + \sqrt{x}\sqrt{y} + x\sqrt{y} + y\sqrt{x}, \qquad x > 0, y > 0$$

Example

Solve

$$(x + ye^{\frac{y}{x}})dx - xe^{\frac{y}{x}}dy = 0,$$

$$y(1) = 0$$

Example

Solve

$$(x + ye^{\frac{y}{x}})dx - xe^{\frac{y}{x}}dy = 0,$$

$$y(1) = 0$$

Solution

$$\sqrt{xy} = x + c$$

Example

Solve

$$y' = \frac{(y - 2x + 1)^2}{y - 2x}$$

Example

Find the value of k so that the given differential equation is exact :

$$(y^3 + kxy^4 - 2x)dx + (3xy^2 + 20x^2y^3)dy = 0$$

Example

Solve

$$y' = 2xy^2 + 3y^2 - 8x - 12$$

$$y(0) = -1$$

Example

Solve the given differential equation by finding an appropriate integrating factor :

$$(y^2 + xy^3)\,dx + (5y^2 - xy + y^3 \sin y)\,dy = 0, \qquad xy \neq 0$$

In Problems 1 - 5 ,solve the given differential equation by separation of variables.

1. * $2x\left(y^2+y\right)\,dx+\left(x^2-1\right)y\,dy=0\,; \quad y\neq 0$
2. * $\left(xy+x\right)\,dx+\left(x^2y^2+x^2+y^2+1\right)\,dy=0$
3. * $\dfrac{dy}{dx}=\dfrac{\sin x+x\cos x}{y\left(2\ln y+1\right)}\,,\; y>0$
4. $\dfrac{dy}{dx}=e^{3x+2y}$
5. $\sec^2 x dy+\csc y dx=0$

6. Solve the initial value problem

$$\begin{cases} y\,dy = 4x\left(y^2+1\right)^{1/2} \\ y\left(0\right) = 1 \end{cases}$$

7. Solve the initial value problem

$$\begin{cases} \left(xy^2+4x\right)\,dx + \left(8y-2x^2y\right)\,dy = 0\,,\ \mid x \mid < 2 \\ y(0) = 0 \end{cases}$$

8. * Solve the initial value problem

$$\begin{cases} \frac{dy}{dx} = 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{xy} \quad , \quad y \neq 0, x \neq 0 \\ y(1) = 1 \end{cases}$$

Equations with Homogeneous Coefficients

In Problems 1 - 5 ,solve the given differential equation

1. $(x - y)\,dx + x\,dy = 0$
2. $y' = \dfrac{y + x}{x}$
3. $(y^2 + yx)\,dx - x^2 dy = 0$
4. $\dfrac{dy}{dx} = \dfrac{x + 3y}{3x + y}$
5. $y' = \dfrac{2xy}{x^2 - y^2}$

6. * Find the general solution of the differential equation

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}; x > 0$$

7. * Solve the initial value problem

$$\begin{cases} (x - y)dx + (3x + y)dy = 0 \\ y(3) = -2 \end{cases}$$

8. * Find the general solution of the differential equation

$$(y - 2\sqrt{xy})dx + xdy = 0, x > 0, y > 0.$$

Solving Some Differential Equations by Using Appropriate Substitutions

In Problems 1 - 4 ,solve the given differential equation by using appropriate substitutions.

1. $\dfrac{dy}{dx} = (-2x + y)^2 - 7.$
2. $\dfrac{dy}{dx} = (x + y + 1)^2.$
3. $\dfrac{dy}{dx} = \dfrac{1 - x - y}{x + y}.$
4. $\dfrac{dy}{dx} = \dfrac{x - y - 3}{x + y - 1}.$

Solving Some Differential Equations by Using Appropriate Substitutions

5. * Obtain the general solution of the differential equation

$$(2x+y)\,\frac{dy}{dx} - 1 - (2x+y)^2 = 0\,, \quad 2x+y \neq 0$$

6. * By using an appropriate substitution, or any other method, find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x}\ln(xy)\ ,\ x>0\,,\ y>0$$

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Exact Equations

In Problems 1 -2 ,determine whether the given differential equation is exact. If it is exact, solve it.

1. $2xy\,dx + \left(1+x^2\right)\,dy = 0$
2. $y' = \dfrac{2+ye^{xy}}{2y-xe^{xy}}$
3. $(\cos y + y\cos x)dx + (\sin x - x\sin y)dy = 0$
4. * Solve the initial value problem

$$\begin{cases} \left(6xy+2y^2-5\right)dx + \left(3x^2+4xy-6\right)dy = 0 \\ y(1) = 1 \end{cases}$$

5. * Find the general solution of the differential equation

$$\left(x+y^2+\sin^{-1}y\right)dx + \left(2xy + \frac{x}{\sqrt{1-y^2}}\right)dy = 0.$$

6. solve the differential equation

$$(2y^2 + 3x)\,dx + 2xydy = 0, \quad x > 0$$

7. solve the differential equation

$$y' = 2xy - x$$

8. * Find m and n such that, $\mu(x, y) = x^m y^n$ will be an integrating factor for the differential equation

$$(5xy^2 - 2y)\,dx + (3x^2y - x)\,dy = 0$$

Then solve the differential equation.