Functions of Random Variables

Two-to Two Transformations. (Joint distribution of Functions of Random Variables

Q1) If $X \sim Uniform(0,1)$ independent of $Y \sim Exponential(1)$, find the distribution of $Z = X + Y$:

a. Using the *pdf* formula derived in class. (Jacobian)

b. By first finding the *cdf* and then differentiating. (convolution)

Solution :

$$X \sim Uniform(0,1) \ \Rightarrow \ f(x) = 1 \quad ,0 < x < 1$$

$$Y \sim Exponential(1) \Rightarrow \ f(y) = e^{-y} \quad , \quad y > 0$$

$$X \text{ and } Y \text{ independent} \Rightarrow f(x,y) = f(x)f(y) = e^{-y} \quad ,0 < x < 1 \quad ,y > 0$$

a)

$$\boldsymbol{Z = X + Y} \ \Rightarrow \ Y = Z - X \ \Rightarrow \ \begin{matrix} \boldsymbol{U = X} \\ Y = Z - U \end{matrix}$$

$$\left.\begin{matrix} 0 < X < 1 \\ 0 < Y < \infty \end{matrix}\right\} \ \Rightarrow \ \begin{matrix} 0 < X + Y < \infty \ \Rightarrow \ 0 < Z < \infty \\ 0 < U < 1 \end{matrix}$$

$$\left.\begin{matrix} 0 < X < 1 \\ 0 < Y < \infty \end{matrix}\right\} \ \Rightarrow \ \begin{matrix} 0 < U < 1 \\ 0 < Z - U < \infty \ \Rightarrow \ U < Z < \infty \end{matrix}$$

$$J(u,z) = \begin{vmatrix} \frac{d}{du}x & \frac{d}{dz}x \\ \frac{d}{du}y & \frac{d}{dz}y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 - 0 = 1 \ \Rightarrow \ |J(u,z)| = 1$$

$$f_{ZU}(z,u) = f_{XY}(x(u,v), y(u,v)) \, |J(u,z)| = \boldsymbol{e^{-(z-u)}}$$

$$f(z) = \int f(z,u)\, du = \begin{cases} \int_0^Z e^{-(z-u)}\, du \ , & ,0 < z < 1 \\ \int_0^1 e^{-(z-u)}\, du \ , & ,1 < z < \infty \end{cases}$$

$$= \begin{cases} 1 - e^{-z} \ , & ,0 < z < 1 \\ e^{-z}(e - 1) \ , & ,1 < z < \infty \end{cases}$$

b) Other solution: By use the convolution - page 320

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\, dy = \int_{-\infty}^{\infty} f_Y(a-x) f_X(x)\, dx$$

$z - x > 0 \Rightarrow x < z \; and \; 0 < x < 1$ so , **if $0 < z < 1$**

$$f_{X+Y}(z) = \int_0^z (e^{-z+x})(1) \; dx = e^{-z}[e^x]_0^z = e^{-z}(e^z - 1) = 1 - e^{-z}$$

If $1 < z < \infty$ then

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_Y(z-x) f_X(x) \; dx = \int_0^1 (e^{-z+x})(1) \; dx = e^{-z}[e^x]_0^1 = e^{-z}(e^1 - 1)$$

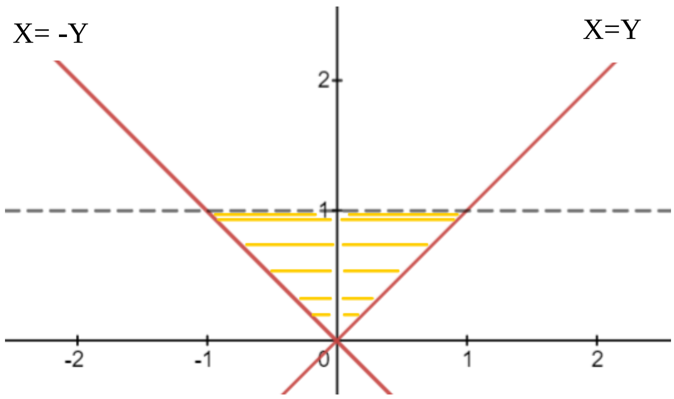
Q2) Let X and Y have joint pdf $f(x, y) = 1 \; ; \; -y < x < y , \; 0 < y < 1.$

a. Find the conditional pdf of $X|Y = y$.

b. Find $P(X < 0|Y = y)$.

c. Find $P(X > \frac{1}{4} \,|Y = \frac{1}{3})$.

d. Find $P(0 < X < \frac{1}{4} \,|Y = \frac{1}{2})$.



Solution : H.W

a) $f(X|Y = y) = \frac{f(x,y)}{f(y)}$

$$f(y) = \int_x f(x, y) \, dx = \int_{-y}^{y} 1 \; dx = [x]_{-y}^{y}$$
$$= y + y = 2y$$

$f(X|Y = y) = \frac{f(x,y)}{f(y)} = \frac{1}{2y}$; $-y < x < y$ (y is fixed value)

b) $P(X < 0|Y = y) = \int_{-y}^{0} \frac{1}{2y} \, dx = \frac{1}{2y}[x]_{-y}^{0} = \frac{1}{2y}(y) = \frac{1}{2}$

c) $P\left(X > \frac{1}{4} \middle| Y = \frac{1}{3}\right) = \int_{1/4}^{y} \frac{1}{2y} \, dx = \int_{1/4}^{1/3} \frac{1}{2\left(\frac{1}{3}\right)} \, dx = \frac{3}{2} \, [x]_{\frac{1}{4}}^{1/3} = \frac{3}{2}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{8}$

d) $P\left(0 < X < \frac{1}{4} \middle| Y = \frac{1}{2}\right) = \int_{0}^{1/4} \frac{1}{2\left(\frac{1}{2}\right)} \, dx = [x]_0^{\frac{1}{4}} = \frac{1}{4}$

Q3) Let X and Y have joint pdf $f(x, y) = \frac{2}{5} (x + 4y) \; ; \; 0 < x < 1, \; 0 < y < 1.$

a. Find the conditional pdf of $Y|X = x$.

b. Find $P\left(Y < \frac{1}{3} \middle| X = \frac{1}{2}\right)$.

Solution :

a) $f(Y|X=x)=\frac{f(x,y)}{f(x)}$

$$f(x)=\int_y f(x,y)\,dy=\int_0^1 \frac{2}{5}(x+4y)\,dy=\frac{2}{5}\left[xy+\frac{4y^2}{2}\right]_0^1=\frac{2}{5}(x+2)$$

$$\therefore f(Y|X=x)=\frac{\frac{2}{5}(x+4y)}{\frac{2}{5}(x+2)}=\frac{x+4y}{x+2}$$

b) $f\left(Y\middle|X=\frac{1}{2}\right)=\frac{\frac{1}{2}+4y}{\frac{1}{2}+2}=\frac{1+8y}{5}$

$$P\left(Y<\frac{1}{3}\middle|X=\frac{1}{2}\right)=\int_0^{\frac{1}{3}} f\left(Y\middle|X=\frac{1}{2}\right)dy$$

$$=\int_0^{1/3}\frac{1+8y}{5}\,dy=\frac{1}{5}\left[y+\frac{8y^2}{2}\right]_0^{\frac{1}{3}}=\frac{1}{5}\left(\frac{1}{3}+\frac{4}{9}\right)=\frac{7}{45}$$

Q4) If $X\sim Uniform(0,1)$ independent of $Y\sim Exponential(1)$, find

a. The joint density function of $Z=X+Y$ and $U=X/Y$.

b. The density function of Z.

c. The density function of U.

Solution :

a) $X\sim Uniform(0,1)\ \Rightarrow\ f(x)=1\quad ,0<x<1$

$Y\sim Exponential(1)\Rightarrow\ f(y)=e^{-y}\quad ,\quad y>0$

$X\ and\ Y\ independent\Rightarrow f(x,y)=f(x)f(y)=\ e^{-y}\quad ,0<x<1\ \ ,y>0$

$$\boldsymbol{Z=X+Y}\ \Rightarrow\ Y=Z-X$$

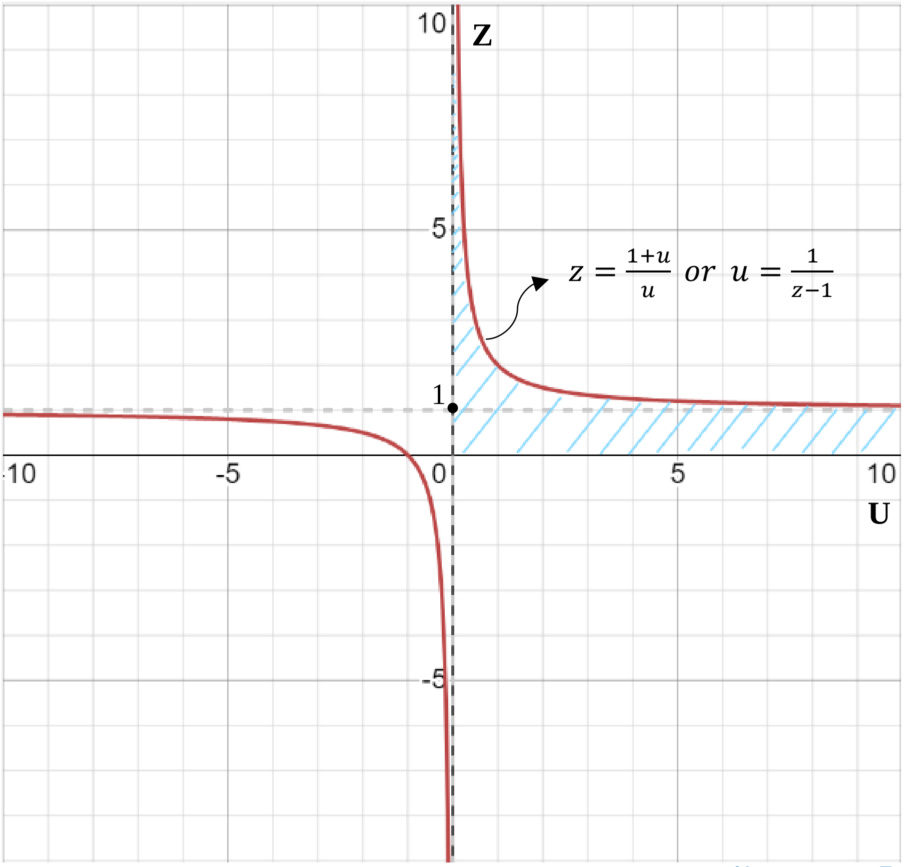
$$U=\frac{X}{Y}\ \Rightarrow\ X=UY=U(Z-X)=UZ-UX\ \Rightarrow\ X+UX=UZ\Rightarrow\ \therefore\ \boldsymbol{X=\frac{UZ}{1+U}}$$

$$Y=Z-X=Z-\frac{UZ}{1+U}=\frac{Z+UZ-UZ}{1+U}=\frac{Z}{1+U}\qquad \therefore \boldsymbol{Y=\frac{Z}{1+U}}$$

$$\left.\begin{array}{l}0<X<1\\ 0<Y<\infty\end{array}\right\}\Rightarrow\quad \begin{array}{l}0<X+Y<(1+\infty)\ \Rightarrow\ 0<Z<\infty\\ 0<\frac{X}{Y}<\infty\quad but\quad 0<U<\infty\end{array}$$

$$\left.\begin{array}{l} 0 < X < 1 \\ 0 < Y < \infty \end{array}\right\} \Rightarrow \begin{array}{l} 0 < \frac{UZ}{1+U} < 1 \;\Rightarrow\; 0 < UZ < 1+U \;\Rightarrow\; 0 < Z < \frac{1+U}{U} \\ \quad\quad \textbf{or} \quad 0 < UZ < 1+U \Rightarrow 0 < U < \frac{1}{Z-1} \end{array}$$

$$0 < \frac{Z}{1+U} < \infty \;\Rightarrow\; 0 < Z < \infty$$



$$\lim_{Z\to\infty} \frac{1}{Z-1} = 0\,,\ \lim_{u\to\infty} \frac{1+u}{u} = 1$$

$$z > 1 \Rightarrow\ 0 < u < \frac{1}{z-1}$$

$$0 < z < 1 \Rightarrow\ 0 < u < \infty$$

$$0 < u < \infty \Rightarrow 0 < z < \frac{u+1}{u}$$

$$J(x,y) = \begin{vmatrix} \frac{d}{dz}\,x & \frac{d}{du}\,x \\ \frac{d}{dz}\,y & \frac{d}{du}\,y \end{vmatrix} = \begin{vmatrix} \frac{u}{1+u} & \frac{z}{(1+u)^2} \\ \frac{1}{1+u} & -\frac{z}{(1+u)^2} \end{vmatrix}$$

$$= \left(\frac{u}{1+u}\right)\left(-\frac{z}{(1+u)^2}\right) - \left(\frac{1}{1+u}\right)\left(\frac{z}{(1+u)^2}\right) = \frac{-z}{(1+u)^3}[u+1] = \frac{-z}{(1+u)^2}$$

$$|\,J(u,z)| = \frac{z}{(1+u)^2}$$

$$f(z,u) = f_{XY}(u,z)\,|\,J(u,z)| = e^{-\left(\frac{z}{1+u}\right)}\ \frac{z}{(1+u)^2} \qquad Z > 0\ , \qquad u > 0\,, 0 < z < \frac{u}{1+u}$$

b) The density function of Z

$$f(z)=\begin{cases}\int_0^{\frac{1}{z-1}}\frac{z}{(1+u)^2}\ e^{-\left(\frac{z}{1+u}\right)}\ du\ =\int_{-z}^{1-z}e^{w}\ dw=e^{1-z}-e^{-z}=e^{-z}(e-1) & ,z>1\\ \int_0^{\infty}\frac{z}{(1+u)^2}\ e^{-\left(\frac{z}{1+u}\right)}\ du\ =\int_{-z}^{0}e^{w}\ \ dw=1-e^{-z} & ,\ 0<z<1\end{cases}$$

Using that $w=-\frac{z}{1+u}\ \Rightarrow\ \ dw=\frac{z}{(1+u)^2}du$

At $u=\ 0\Rightarrow w=-\frac{z}{1+0}=\ -z$

At $u=\frac{1}{z-1}\Rightarrow w=\frac{-z}{1+(\frac{1}{z-1})}=\frac{-z}{\frac{z-1+1}{z-1}}=\ -\frac{z(z-1)}{z}=-(Z-1)=1-z$

At $u=\infty\Rightarrow w=-\frac{z}{1+\infty}=\ 0$

c) The density function of U

$$f(u)=\int_0^{\frac{u+1}{u}}\frac{z}{(1+u)^2}\ e^{-\left(\frac{z}{1+u}\right)}\ dz=\frac{1}{(1+u)^2}\int_0^{\frac{u+1}{u}}z\,e^{-\left(\frac{z}{1+u}\right)}dz$$

By use Integration by Parts $\int_a^b u\,dv=[uv]_a^b-\int_a^b v\,du$

Let $u=z\qquad dv=e^{-\left(\frac{z}{u+1}\right)}\ \ \Rightarrow\ \ du=dz\qquad v=-(u+1)\,e^{-\left(\frac{z}{u+1}\right)}$

$$f(u)=\frac{1}{(1+u)^2}\left[-z(u+1)\,e^{-\frac{z}{u+1}}\right]_0^{\frac{u+1}{u}}-\frac{1}{(1+u)^2}\int_0^{\frac{u+1}{u}}-(u+1)\,e^{-\left(\frac{z}{1+u}\right)}dz$$

$$=\frac{1}{(1+u)^2}\left[-\left(\frac{u+1}{u}\right)(u+1)e^{-\frac{u+1}{u(u+1)}}\right]-\int_0^{\frac{u+1}{u}}-\left(\frac{1}{u+1}\right)e^{-\left(\frac{z}{1+u}\right)}dz$$

$$-\frac{1}{u}e^{-\frac{1}{u}}-\left[e^{-\frac{z}{u+1}}\right]_0^{\frac{u+1}{u}}=-\frac{1}{u}e^{-\frac{1}{u}}-\left[e^{-\frac{1}{u}}-e^{0}\right]=-\frac{1}{u}e^{-\frac{1}{u}}-e^{-\frac{1}{u}}+1$$

$$f(u)=1-e^{-\frac{1}{u}}\left(\frac{1}{u}+1\right)\quad ;\ u>0$$

Note: $\frac{A}{0}=\infty\,,\ \ \frac{A}{\infty}=0\,,\ \ \frac{0}{A}=0$

Q5) Let (X,Y) have joint pdf $f(x,y)=\frac{1}{x^2\,y^2}\quad ;x\geq 1\ ,y\geq 1$

a. Find the joint density of $U=XY$ and $V=X/Y$.

b. What are the marginal density of U and V?

Solution :

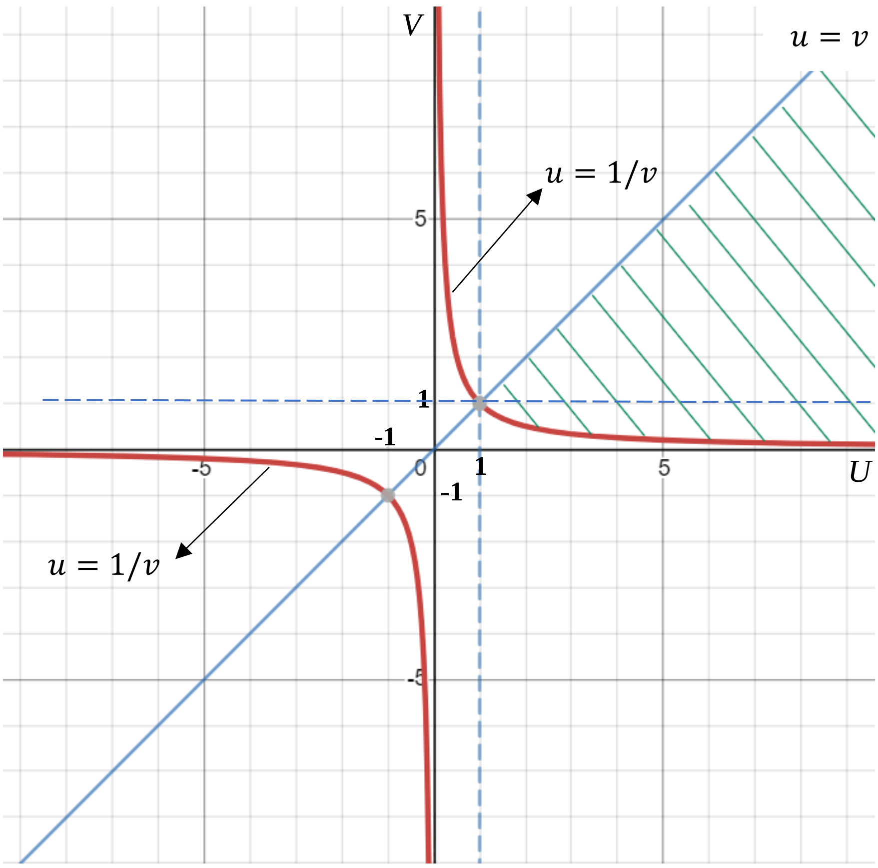
a) $U = XY \;\Rightarrow\; Y = \frac{U}{X}$

$$V = \frac{X}{Y} \;\Rightarrow\; VY = X \Rightarrow\; V\frac{U}{X} = X \Rightarrow\; X^2 = VU \;\Rightarrow\; \therefore\; \boldsymbol{X = \sqrt{VU}}$$

$$Y = \frac{U}{X} \;\Rightarrow\; Y = \frac{U}{\sqrt{VU}} \;\Rightarrow\; \therefore\, \boldsymbol{Y = \sqrt{\frac{U}{V}}}$$

$$\left.\begin{matrix} x > 1 \\ y > 1 \end{matrix}\right\} \Rightarrow \quad \begin{matrix} xy > 1 \Rightarrow & u > 1 \\ \frac{x}{y} > 0 \Rightarrow & v > 0 \end{matrix}$$

$$\left.\begin{matrix} x > 1 \\ y > 1 \end{matrix}\right\} \Rightarrow \quad \begin{matrix} \sqrt{vu} > 1 \Rightarrow \quad vu > 1 \;\Rightarrow u > \frac{1}{v} \\ \sqrt{\frac{u}{v}} > 1 \Rightarrow \quad \frac{u}{v} > 1 \;\Rightarrow\; u > v \end{matrix}$$



$$\lim_{v\to\infty} \frac{1}{v} = 0$$

$$\frac{1}{u} < v < u \;\Rightarrow\; 1 < u < \infty$$

$$\begin{cases} \frac{1}{v} < u < \infty \;\Rightarrow\; 0 < v < 1 \\ v < u < \infty \;\;\Rightarrow\; 1 < v < \infty \end{cases}$$

$$J(u,v) = \begin{vmatrix} \frac{d}{du}x & \frac{d}{dv}x \\ \frac{d}{du}y & \frac{d}{dv}y \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \\ \frac{1}{2\sqrt{vu}} & \frac{-\sqrt{u}}{2v^{\frac{3}{2}}} \end{vmatrix} = \frac{\sqrt{v}}{2\sqrt{u}}\left(\frac{-\sqrt{u}}{2v^{\frac{3}{2}}}\right) - \frac{1}{2\sqrt{vu}}\left(\frac{\sqrt{u}}{2\sqrt{v}}\right)$$

$$= \frac{-1}{4v} - \frac{1}{4v} = -\frac{1}{2\,v}$$

$$|J(u,v)| = \frac{1}{2\,v}$$

$$f_{UV}(u,v) = f_{XY}(u,v)|J(u,v)| = \frac{1}{(uv)\left(\frac{\boldsymbol{u}}{v}\right)}\ \frac{1}{2v} = \frac{\mathbf{1}}{\mathbf{2\,u^2\,v}} \quad ;u>1\,,u>\frac{1}{v}\,,0<\ v<u$$

b) marginal density of U and V

$$f(u) = \int_{\frac{1}{u}}^{u} \frac{1}{2\,u^2\,v}\ dv = \frac{1}{2u^2}\int_{\frac{1}{u}}^{u} \frac{1}{v}\ dv = \frac{1}{2u^2}\ [\ln v]_{\frac{1}{u}}^{u} = \frac{1}{2u^2}\left(\ln u - \ln\frac{1}{u}\right)$$

$$= \frac{1}{2u^2}(\ln u - \ln 1 + \ln u) = \frac{1}{2u^2}\ (2\ln u - 0)$$

$$\boldsymbol{f(u) = \frac{1}{u^2}\ln u} \quad ,\mathbf{1<u<\infty}$$

$$f(v) = \begin{cases} \displaystyle\int_{\frac{1}{v}}^{\infty} \frac{1}{2\,u^2\,v}\ du = \frac{1}{2v}\int_{\frac{1}{v}}^{\infty} \frac{1}{u^2}\ du = -\frac{1}{2v}\left[\frac{1}{u}\right]_{\frac{1}{v}}^{\infty} = \frac{\mathbf{1}}{\mathbf{2}} & ,\mathbf{0<v<1} \\ \displaystyle\int_{v}^{\infty} \frac{1}{2\,u^2\,v}\ du = \frac{1}{2v}\int_{v}^{\infty} \frac{1}{u^2}\ du = -\frac{1}{2v}\left[\frac{1}{u}\right]_{v}^{\infty} = \frac{\mathbf{1}}{\mathbf{2v^2}} & ,\ \mathbf{1<v<\infty} \end{cases}$$

Q6) Let X_1 and X_2 be independent $Exp(\lambda)$ r.v. Find the joint density of

$Y_1 = X_1 + X_2$ *and* $Y_2 = e^{X_1}$.

Solution :

$$X_1 \sim Exp(\lambda) \Rightarrow\ f(x_1) = \lambda\, e^{-\lambda x_1} \qquad ,x_1>0$$

$$X_2 \sim Exp(\lambda) \Rightarrow\ f(x_2) = \lambda\, e^{-\lambda x_2} \qquad ,x_2>0$$

$$X_1 \text{ and } X_2 \text{ independent} \Rightarrow f(x_1,x_2) = f(x_1)f(x_2) = \lambda^2\, e^{-\lambda(x_1+x_2)} \qquad ,x_1>0\,,x_2>0$$

$$Y_1 = X_1 + X_2 \Rightarrow\ X_2 = Y_1 - X_1$$

$$Y_2 = e^{X_1} \Rightarrow\ \therefore \boldsymbol{X_1 = \ln Y_2}$$

$$X_2 = Y_1 - X_1 = Y_1 - \ln Y_2\ \Rightarrow\ \ \therefore \boldsymbol{X_2 = Y_1 - \ln Y_2}$$

$$\left.\begin{matrix} 0<x_1<\infty \\ 0<x_2<\infty \end{matrix}\right\} \Rightarrow \begin{matrix} 0<x_1+x_2<\infty \Rightarrow\ \ 0<y_1<\infty \\ 1<e^{x_1}<\infty\ \Rightarrow\ \ 1<y_2<\infty \end{matrix}$$

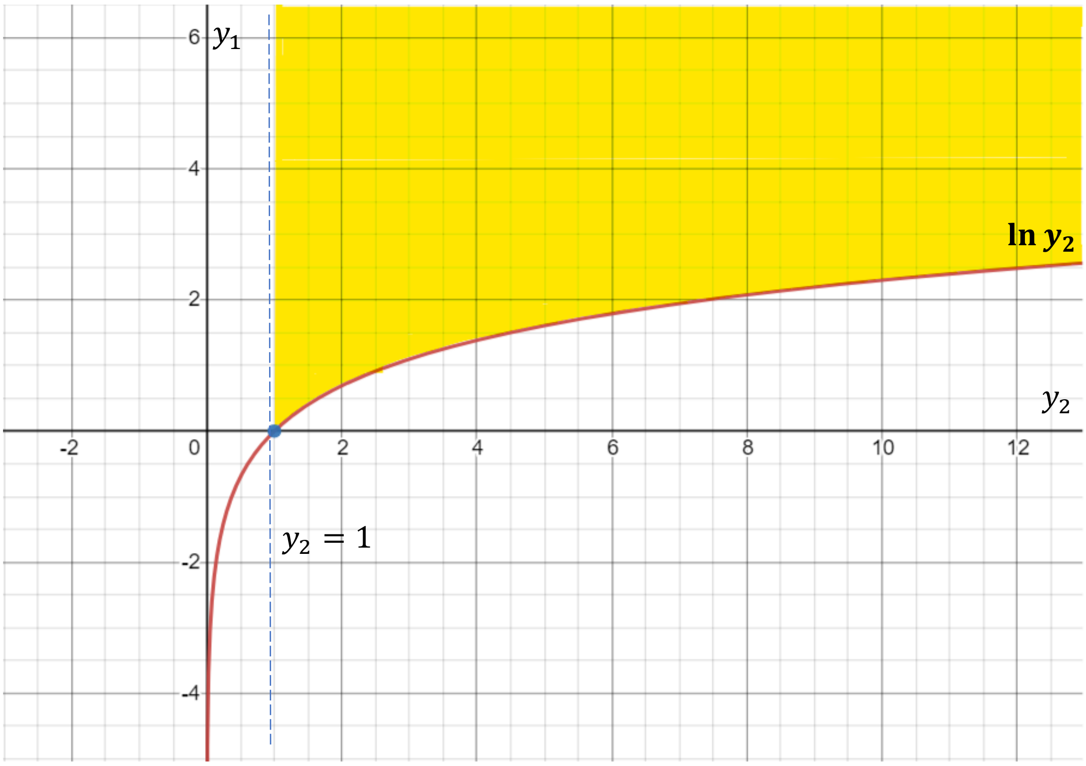
$$\left.\begin{matrix} 0<x_1<\infty \\ 0<x_2<\infty \end{matrix}\right\} \Rightarrow \begin{matrix} 0<\ln y_2\ <\infty \\ 0<y_1-\ln y_2<\infty\ \Rightarrow\ \ \ln y_2<y_1<\infty \end{matrix}$$

$$J(x_1,x_2) = \begin{vmatrix} \frac{d}{dy_1}x_1 & \frac{d}{dy_2}x_1 \\ \frac{d}{dy_1}x_2 & \frac{d}{dy_2}x_2 \end{vmatrix} = \begin{vmatrix} 0 & 1/y_2 \\ 1 & -1/y_2 \end{vmatrix} = \frac{-1}{y_2}$$

$$|J(y_1,y_2)| = \frac{1}{y_2}$$

$$f(y_1,y_2)=f_{X_1X_2}(y_1,y_2)|J(y_1,y_2)|=\ \lambda^2\ e^{-\lambda\,(\ln y_2+y_1-\ln y_2)}\ \frac{1}{y_2}$$

$$\boldsymbol{f(y_1,y_2)=\frac{\lambda^2}{y_2}\ e^{-\lambda\,y_1}\quad ;\ 0<\ln y_2<y_1\ \ ,\ \ y_2>1}$$



Q7) Let $X_1 \sim Exp(\lambda_1)$ independent of $X_2 \sim Exp(\lambda_2)$ r.v.. Find:

a. The cumulative distribution function of $Z=\frac{x_1}{x_2}$

b. $P(X_1<X_2)$.

Solution :

$X_1 \sim Exp(\lambda_1) \Rightarrow\ f(x_1)=\lambda_1\ e^{-\lambda_1 x_1}\qquad ,x_1>0$

$X_2 \sim Exp(\lambda_2) \Rightarrow\ f(x_2)=\lambda_2\ e^{-\lambda_2 x_2}\qquad ,x_2>0$

a) X_1 *and* X_2 *independent*

$$\boldsymbol{\Rightarrow f(x_1,x_2)=f(x_1)f(x_2)=\lambda_1\lambda_2\ e^{-(\lambda_1x_1+\lambda_2x_2)}\qquad ,x_1>0\,,x_2>0}$$

$$Z=\frac{X_1}{X_2}\ \Rightarrow\ X_2=\frac{X_1}{Z}=\frac{U}{Z}\ \Rightarrow\ \ \therefore\ \boldsymbol{X_2=\frac{U}{Z}}$$

$$U=X_1\ \Rightarrow\ \ \therefore\ \boldsymbol{X_1=U}$$

$$\left.\begin{matrix}0<x_1<\infty\\0<x_2<\infty\end{matrix}\right\}\Rightarrow\ \begin{matrix}0<u<\infty\\0<z<\infty\end{matrix}$$

$$\left.\begin{matrix} 0 < x_1 < \infty \\ 0 < x_2 < \infty \end{matrix}\right\} \Rightarrow \begin{matrix} 0 < u < \infty \\ 0 < \frac{u}{z} < \infty \end{matrix} \quad \Rightarrow \quad u > 0\,,\ z > 0$$

$$J(u,z) = \begin{vmatrix} \frac{d}{du}\,x_1 & \frac{d}{dz}\,x_1 \\ \frac{d}{du}\,x_2 & \frac{d}{dz}\,x_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1/z & -u/z^2 \end{vmatrix} = -\frac{u}{z^2} \quad \Rightarrow \quad |\,J(u,z)| = \frac{u}{z^2}$$

$$f(z,u) = f_{X_1X_2}(u,z)|\,J(u,z)| = \lambda_1\lambda_2\, e^{-\left(\lambda_1 u + \lambda_2 \frac{u}{z}\right)}\, \frac{u}{z^2}$$

$$\boldsymbol{f(z,u) = \lambda_1\lambda_2 \frac{u}{z^2}\, e^{-u\left(\lambda_1 + \lambda_2 \frac{1}{z}\right)} \quad ;\ u > 0, \qquad z > 0}$$

$$f(z) = \int_0^\infty f(z,u)\, du = \lambda_1\lambda_2 \frac{1}{z^2} \int_0^\infty u\, e^{-u\left(\lambda_1 + \lambda_2 \frac{1}{z}\right)}\, du$$

$\because \int_0^\infty x^a\, e^{-b\,x}\, dx = \frac{\Gamma(a+1)}{b^{a+1}}$, $\Gamma(a) = (a-1)!$

$$= \lambda_1\lambda_2 \frac{1}{z^2} \frac{\Gamma(2)}{\left(\lambda_1 + \lambda_2 \frac{1}{z}\right)^2} = \lambda_1\lambda_2 \frac{1}{z^2} \frac{\Gamma(2)}{\left(\frac{\lambda_1 z + \lambda_2}{Z}\right)^2}$$

$$\boldsymbol{f(z) = \frac{\lambda_1\lambda_2}{(\lambda_1 z + \lambda_2)^2} \qquad , z > 0}$$

$$F(z) = P(Z \le z) = \int_0^z \frac{\lambda_1\lambda_2}{(\lambda_1 t + \lambda_2)^2}\, dt = \lambda_1\lambda_2 \int_0^z (\lambda_1 t + \lambda_2)^{-2}\, dt$$

$$= \left[\lambda_2 \frac{(\lambda_1 t + \lambda_2)^{-1}}{-1}\right]_0^z = \left[\frac{-\lambda_2}{\lambda_1 t + \lambda_2}\right]_0^z = \frac{-\lambda_2}{(\lambda_1 z + \lambda_2)} + 1 = \boldsymbol{\frac{\lambda_1\, z}{(\lambda_1 z + \lambda_2)}}$$

$$F(z) = \begin{cases} 0 & , z < 0 \\ \dfrac{\lambda_1\, z}{(\lambda_1 z + \lambda_2)} & , z > 0 \end{cases}$$

b) $P(X_1 < X_2) = \int_0^\infty \int_0^{x_2} f(x_1,x_2)\ dx_1\, dx_2 =$

$$\int_0^\infty \int_0^{x_2} \lambda_2\, e^{-\lambda_2 x_2}\, \lambda_1 e^{-\lambda_1 x_1}\ dx_1\, dx_2$$

$$= \int_0^\infty \lambda_2 e^{-(\lambda_2 x_2)} \left[-e^{-(\lambda_1 x_1)}\right]_0^{x_2}\ dx_2$$

$$= \int_0^\infty \lambda_2 e^{-(\lambda_2 x_2)} \left(1 - e^{-\lambda_1 x_2}\right)\ dx_2$$

$$= \int_0^\infty \lambda_2 e^{-(\lambda_2 x_2)}\ dx_2 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \int_0^\infty (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2) x_2}\ dx_2$$

$$= \left[-e^{-(\lambda_2 x_2)}\right]_0^\infty - \frac{\lambda_2}{\lambda_1 + \lambda_2} \left[-e^{-(\lambda_2 x_2)}\right]_0^\infty = 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2}\ ;\ \{e^{-\infty} = 0, e^0 = 1\}$$

Q8) Let X and Y be distributed as independent $Uniform(0,1)$ r.v.

a. Find the joint density function of $Z_1 = X + Y$ and $Z_2 = Y$.

b. Find the marginal pdf of Z_1 from the joint density.

Solution : HW

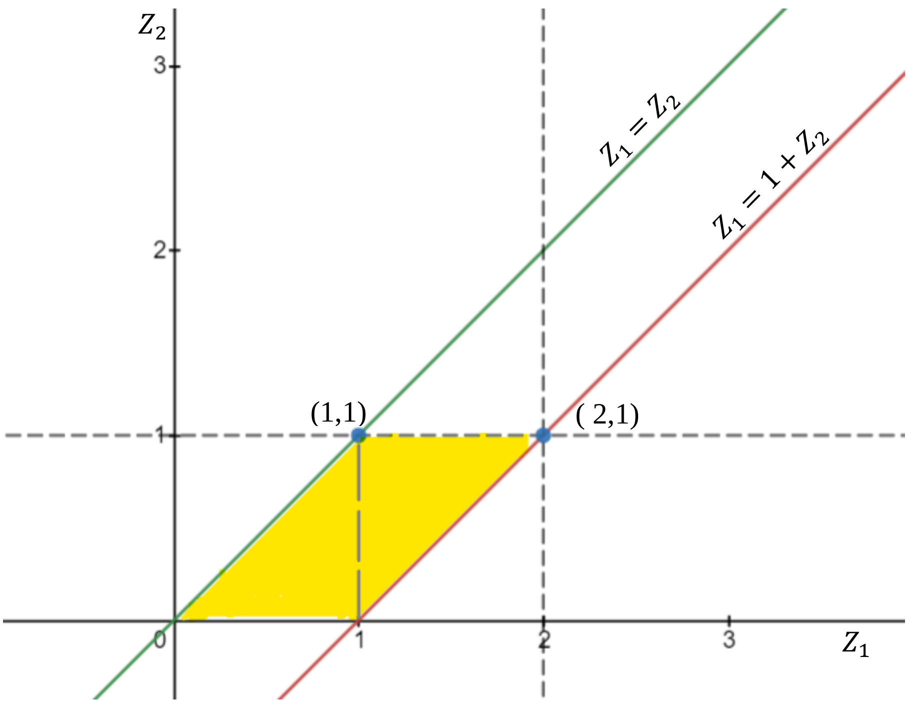
$$Uniform(0,1) \Rightarrow \quad f(x) = 1 \quad ,0 < x < 1$$

a) $X \text{ and } Y \text{ independent } \Rightarrow f(x,y) = f(x)f(y) = 1$

$$Z_1 = X + Y \quad \Rightarrow X = Z_1 - Y \quad \Rightarrow \therefore \boldsymbol{X = Z_1 - Z_2}$$
$$\boldsymbol{Z_2 = Y}$$

$$\left.\begin{matrix} 0 < x < 1 \\ 0 < y < 1 \end{matrix}\right\} \Rightarrow \quad \begin{matrix} 0 < x + y < 2 \;\Rightarrow\; 0 < z_1 < 2 \\ 0 < y < 1 \;\Rightarrow\; 0 < z_2 < 1 \end{matrix}$$

$$\left.\begin{matrix} 0 < x < 1 \\ 0 < y < 1 \end{matrix}\right\} \Rightarrow \quad \begin{matrix} 0 < z_1 - z_2 < 1 \Rightarrow \; z_2 < z_1 < 1 + z_2 \\ or \quad \Rightarrow 1 - z_1 < z_2 < z_1 \\ 0 < z_2 < \infty \end{matrix}$$



$$J(x,y) = \begin{vmatrix} \frac{d}{dz_1}x & \frac{d}{dz_2}x \\ \frac{d}{dz_1}y & \frac{d}{dz_2}y \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \quad ; \; |J(z_1,z_2)| = 1$$

$$f(z_1,z_2) = f_{XY}(z_1,z_2)|\,J(z_1,z_2)| = 1$$

b) marginal pdf of Z_1

$$f(z_1) = \begin{cases} \int_0^{z_1} 1 \; dz_2 = z_1 & ,0 < z_1 < 1 \\ \int_{z_1-1}^{1} 1 \; dz_2 = 2 - z_2 & ,1 < z_1 < 2 \end{cases}$$

Additional part: marginal pdf of Z_2

$$f(z_2) = \int_{z_2}^{1+z_2} 1 \; dz_1 = 1 + z_2 - z_2 = 1 \qquad ,0 < z_2 < 1$$

Q9) Let X and Y be distributed as independent Exp(1) r.v., find:

a. The joint density function of $Z = X + Y$ and $U = \frac{X}{X+Y}$.

b. Find the marginal pdf of U.

Solution :

a) $X \sim Exponential(1) \Rightarrow f(x) = e^{-x} \quad ,x > 0$

$Y \sim Exponential(1) \Rightarrow f(y) = e^{-y} \quad , \quad y > 0$

$X \text{ and } Y \text{ independent} \Rightarrow f(x,y) = f(x)f(y) = e^{-(x+y)} \quad ,x > 0 \;, y > 0$

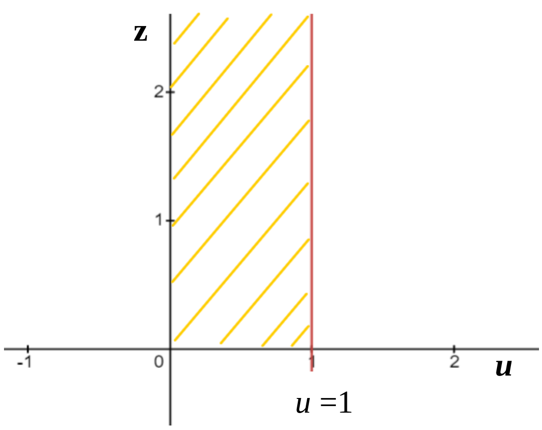
$Z = X + Y \Rightarrow X = Z - Y$

$U = \frac{X}{X+Y} \Rightarrow \quad U = \frac{Z-Y}{Z} \Rightarrow \quad UZ = Z - Y \Rightarrow \quad Y = Z - UZ \Rightarrow \quad \therefore \boldsymbol{Y = Z(1-U)}$

$X = Z - Y \Rightarrow \quad X = Z - Z(1-U) \Rightarrow \quad \therefore \boldsymbol{X = ZU}$

$$\left.\begin{matrix} x > 0 \\ y > 0 \end{matrix}\right\} \Rightarrow \quad \begin{matrix} x + y > 0 \Rightarrow z > 0 \\ \frac{x}{x+y} > 0 \text{ and } x + y > x \Rightarrow 0 < u < 1 \end{matrix}$$

$$\left.\begin{matrix} x > 0 \\ y > 0 \end{matrix}\right\} \Rightarrow \quad \begin{matrix} zu > 0 \Rightarrow z > 0 \text{ or } u > 0 \\ z(1-u) > 0 \Rightarrow z > 0 \text{ or } (1-u) > 0 \Rightarrow z > 0 \text{ or } u < 1 \end{matrix}$$



$$J(x,y)=\begin{vmatrix}\frac{d}{dz}\,x & \frac{d}{du}\,x\\ \frac{d}{dz}\,y & \frac{d}{du}\,y\end{vmatrix}=\begin{vmatrix}u & z\\ 1-u & -z\end{vmatrix}=-zu-z(1-u)=-z$$

$$|J(z,u)|=z$$

$$f(z,u)=f_{XY}(z,u)\,|J(z,u)|=z\,e^{-(zu+z(1-u))}=z\,e^{-z}\quad ;\ z>0\,,0<\ u<1$$

b) Marginal pdf of U

$$f(u)=\int_0^{\infty} z\,e^{-z}\ dz\ =\frac{\Gamma(2)}{1^2}=1\ \ ,0<u<1$$

Additional part : marginal pdf of Z is $f(z)=\int_0^1 z\,e^{-z}\ du\ =z\,e^{-z}$

Q10) Let X and Y have independent $Gamma(\alpha,\lambda)$ distributions.

a. Find the joint pdf of $U\ =\frac{X}{X+Y}$ and $V\ =\ X\ +\ Y$.

b. Show that the marginal density of U is a *Beta distribution*.

Solution : H.W

$$X\sim Gamma(\alpha,\lambda)\Rightarrow f(x)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\,x^{\alpha-1}\,e^{-\lambda\,x}\qquad ,x>0$$

$$Y\sim Gamma(\alpha,\lambda)\Rightarrow f(y)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\,y^{\alpha-1}\,e^{-\lambda\,y}\qquad ,y>0$$

a) $X\ and\ Y\ independent\Rightarrow$

$$f(x,y)=f(x)f(y)=\left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)^2\,x^{\alpha-1}\,y^{\alpha-1}\,e^{-\lambda\,(x+y)}\quad ,x>0\ ,y>0$$

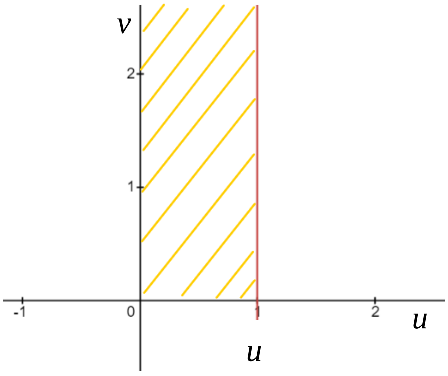
$$V=X+Y\ \Rightarrow\ X=V-Y$$

$$U=\frac{X}{X+Y}\Rightarrow\quad U=\frac{V-Y}{V}\Rightarrow\quad UV=V-Y\ \Rightarrow\quad Y=V-UV\ \Rightarrow\quad \therefore\ \boldsymbol{Y=V(1-U)}$$

$$V=X+Y\ \Rightarrow\ \therefore\ \boldsymbol{X=VU}$$

$$\left.\begin{matrix}x>0\\ y>0\end{matrix}\right\}\Rightarrow\quad \begin{matrix}x+y>0\ \Rightarrow\quad v>0\\ \frac{x}{x+y}>0\quad\Rightarrow\quad 0<u<1\end{matrix}$$

$$\left.\begin{matrix}x>0\\ y>0\end{matrix}\right\}\Rightarrow\quad \begin{matrix}vu>0\ \Rightarrow\quad v>0\quad or\ \ u>0\\ v(1-u)>0\Rightarrow\quad v>0\ or\ \ (1-u)>0\ \Rightarrow\quad v>0\ \ or\ \ u<1\end{matrix}$$



$$J(u,v)=\begin{vmatrix}\frac{d}{du}x & \frac{d}{dv}x\\ \frac{d}{du}y & \frac{d}{dv}y\end{vmatrix}=\begin{vmatrix}v & u\\ -v & 1-u\end{vmatrix}=v(1-u)+vu\ =v$$

$$|J(u,v)|=v$$

$$f(v,u)=f(x,y)\,|J(u,v)|=\left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)^{2}(vu)^{\alpha-1}\big(v(1-u)\big)^{\alpha-1}e^{-\lambda(v)}\quad v$$

$$f(v,u)=\left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)^{2}v^{2\alpha-1}\quad u^{\alpha-1}(1-u)^{\alpha-1}\ e^{-\lambda v}\qquad ;v>0\,,0<u<1$$

b) $f(u)=\int f(v,u)\,dv$

$$f(u)=\left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)^{2}u^{\alpha-1}(1-u)^{\alpha-1}\int_{0}^{\infty}v^{2\alpha-1}\ e^{-\lambda v}\ dv$$

$\because \int_0^\infty x^a\,e^{-b\,x}\ dx=\frac{\Gamma(a+1)}{b^{a+1}}\quad ,\ \Gamma(a)=(a-1)!$

$$=\left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)^{2}u^{\alpha-1}(1-u)^{\alpha-1}\quad\frac{\Gamma(2\alpha)}{\lambda^{2\alpha}}=\frac{\Gamma(2\alpha)}{\big(\Gamma(\alpha)\big)^{2}}\ u^{\alpha-1}(1-u)^{\alpha-1}$$

$$=\frac{\Gamma(\alpha+\alpha)}{\Gamma(\alpha)\Gamma(\alpha)}\ u^{\alpha-1}(1-u)^{\alpha-1}\qquad ,0<u<1$$

Note (unimportant):

$X\sim Beta(a,b)\Rightarrow f(x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ x^{a-1}(1-x)^{b-1}\quad ,0<x<1$

$$\therefore\ U\sim Beta(\alpha,\alpha)$$

Summary of Differentiation Rules:

Power rule:
if $f(x) = x^n$ then $f'(x) = nx^{n-1}$

Sum rule:
if $f(x) = g(x) + h(x)$ then $f'(x) = g'(x) + h'(x)$

Product rule:
if $f(x) = g(x)h(x)$ then $f'(x) = g'(x)h(x) + g(x)h'(x)$

Quotient rule:
if $f(x) = \frac{g(x)}{h(x)}$ then $f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}$

if $f(x) = \sqrt{u}$ $then$ $f'(x) = \frac{u'}{2\sqrt{u}}$

$$example\ f(x)\sqrt{x} = x^{\frac{1}{2}}\ then\ f'(x) = \frac{1}{2}\, x^{\frac{1}{2}-1} = \frac{1}{2}\, x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Exponential and Logarithm Functions:

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln(a) \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Summary of Integration Rules

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C; \quad n \neq -1 \qquad \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C; \quad n \neq -1$$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C \qquad \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int e^x\,dx = e^x + C \qquad \int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$$

Natural Log Rules

$\ln 1 = 0$
$\ln 0 = -\infty$
$\ln \infty = 0$
$\ln e = 1$, $\ln e^x = x$

Product property	$\ln ab = \ln a + \ln b$
Quotient property	$\ln \frac{a}{b} = \ln a - \ln b$
Power property	$\ln m^{p} = p \ln m$
Exponential & Logarithmic	$e^{\ln x} = x$
Inverse property	$\ln e^{x} = x \quad \text{for } x > 0$

$e^{0} = 1$, $e^{-\infty} = 0$, $e^{\infty} = \infty$