i. JOINT, MARGINAL, AND CONDITIONAL DISTRIBUTIONS
ii. COVARIANCE, CORRELATION, INDEPENDENCE OF VARIABLES (STOCHASTIC INDEPENDENCE)

Q1) The joint probability function of two discrete random variables X and Y is given by $f(x, y) = cxy$ for $x = 1, 2, 3$ and $y = 1, 2, 3$ and equals zero otherwise. Find:

a. The constant c.
b. $P(X = 2, Y = 3)$.
c. $P(1 \leq X \leq 2, Y \leq 2)$.
d. $P(X \geq 2)$.
e. $P(Y < 2)$.
f. $P(X = 1)$.
g. $P(Y = 3)$.

Solution :

$$f(x, y) = cxy \quad ; x = 1,2,3 \ ; y = 1,2,3$$

a) We know that $\sum_x \sum_y f(x, y) = 1$

$\sum_{i=1}^{3} \sum_{j=1}^{3} f(x_i, y_j) = 1$

$C + 2C + 3C + 2C + 4C + 6C + 3C + 6C + 9C = 1 \implies C = \frac{1}{36}$

Or $\quad 6C + 12C + 18C = 1 \implies C = \frac{1}{36}$

$$f(x, y) = \frac{1}{36} xy \quad ; x = 1,2,3 \ ; y = 1,2,3$$

Y \ X	1	2	3	$f_Y(y) = \sum_x f(x, y)$
1	1C	2C	3C	6C
2	2C	4C	6C	12C
3	3C	6C	9C	18C
$f_X(x) = \sum_y f(x, y)$	6C	12C	18C	1

b) $P(X = 2, Y = 3) = f(2,3) = 6C = \frac{6}{36} = \frac{1}{6}$

c) $P(1 \leq X \leq 2, Y \leq 2) \qquad ; x = 1,2 \ , y = 1,2$

$P(1 \leq X \leq 2, Y \leq 2) = P(X = 1, Y = 1) + P(X = 1, Y = 2)$

$\qquad\qquad\qquad\qquad + P(X = 2, Y = 1) + P(X = 2, Y = 2)$

$\qquad\qquad\qquad\qquad = f(1,1) + f(1,2) + f(2,1) + f(2,2)$

$\qquad\qquad\qquad\qquad = \frac{1}{36}[1 + 2 + 2 + 4] = \frac{9}{36}$

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d) $P(X \geq 2)$; $x = 2,3$

$$P(X \geq 2) = f_X(2) + f_X(3) = \frac{12}{36} + \frac{18}{36} = \frac{30}{36}$$

e) $P(Y < 2)$; $y = 1$

$$P(Y < 2) = f_Y(1) = \frac{6}{36} = \frac{1}{6}$$

f) $P(X = 1) = f_X(1) = \frac{6}{36} = \frac{1}{6}$

g) $P(Y = 3) = f_Y(3) = \frac{18}{36}$

Q2) For the random variables of **Problem 1**, find the marginal probability function of X and Y. Determine whether X and Y are independent.

Solution :

Marginal dis of X :

x	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	total
$f_X(x) = P(X = x) = \sum_y f(x,y)$	6/36 = 1/6	12/36 =2/6	18/36 =3/6	1

Marginal dis of Y :

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_Y(y)$	6/36	12/36	18/36	1

Are X and Y independent ?

If $f_{XY}(x,y) = f_X(x) \, f_Y(y)$ $\forall$ $x = 1,2,3$; $y = 1,2,3$
Then X and Y are independent. But if there some values of (x) and (y) which make that $f_{XY}(x,y) \neq f_X(x) \, f_Y(y)$ then X and Y are **not** independent .

In this example, we have :

$$f(1,1) = f_X(1) f_Y(1) \Rightarrow \frac{1}{36} = \left(\frac{6}{36}\right)\left(\frac{6}{36}\right)$$

$$f(1,2) = f_X(1) f_Y(2) \Rightarrow \frac{2}{36} = \left(\frac{6}{36}\right)\left(\frac{12}{36}\right)$$

$$\vdots$$

$$f(3,3) = f_X(3) f_Y(3) \Rightarrow \frac{9}{36} = \left(\frac{18}{36}\right)\left(\frac{18}{36}\right)$$

So as $f(x,y) = f_X(x) f_Y(y)$ $\forall$ $x = 1,2,3$; $y = 1,2,3$, then X and Y are independent.

Q3) For the distribution of **Problem 1**, find the **conditional** probability function of X given Y, Y given X.

Solution :

Distribution of $X|Y: f_{X|Y}(x) = \frac{f(x,y)}{f_y(y)}$; $x = 1, 2, 3$

If $Y = 1$: $f_{X|Y=1}(x) = \frac{f(x,1)}{f_y(1)} = \frac{f(x,1)}{6/36}$

x	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	total
$f_{X|Y=1}(y)$	$\frac{f(1,1)}{6/36}$ $= \frac{1/36}{6/36}$ $= 1/6$	$\frac{f(2,1)}{6/36}$ $= \frac{2/36}{6/36} = \frac{2}{6}$ $= 1/3$	$\frac{f(3,1)}{6/36} = \frac{3}{6}$ $= 1/2$	1

If $Y = 2$: $f_{X|Y=2}(x) = \frac{f(x,2)}{f_y(2)} = \frac{f(x,2)}{12/36}$

x	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	total
$f_{X|Y=2}(y)$	$\frac{f(1,2)}{12/36}$ $= \frac{2/36}{12/36}$ $= 1/6$	$\frac{f(2,2)}{12/36}$ $= \frac{4/36}{12/36}$ $= 2/6$	$\frac{f(3,2)}{12/36}$ $= \frac{6/36}{12/36}$ $= 3/6$	1

If $Y = 3$: $f_{X|Y=2}(x) = \frac{f(x,3)}{f_y(3)} = \frac{f(x,3)}{18/36}$

x	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	total
$f_{X|Y=3}(y)$	$\frac{f(1,3)}{18/36}$ $= \frac{3/36}{18/36}$ $= 1/6$	$\frac{f(2,3)}{18/36}$ $= \frac{6/36}{18/36}$ $= 2/6$	$\frac{f(3,3)}{18/36}$ $= \frac{9/36}{18/36}$ $= 3/6$	1

Note: Since X and Y are independent so $f_{X|Y}(x) = \frac{f(x,y)}{f_y(y)} = \frac{f_X(x)f_Y(y)}{f_y(y)} = f_X(x) = \frac{x}{6}$

Distribution of $Y|X$: $f_{Y|X}(y) = \frac{f(x,y)}{f_X(x)}$; $y = 1, 2, 3$

If $X = 1$: $f_{Y|X=1}(y) = \frac{f(1,y)}{f_X(1)} = \frac{f(1,y)}{6/36}$ $\Longrightarrow$

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_{Y|X=1}(y)$	$\dfrac{f(1,\,1)}{6/36}$ $=\dfrac{1/36}{6/36}$ $=\mathbf{1/6}$	$\dfrac{f(1,\,2)}{6/36}$ $=\dfrac{2/36}{6/36}$ $=\mathbf{2/6}$	$\dfrac{f(1,\,3)}{6/36}$ $=\dfrac{3/36}{6/36}$ $=\mathbf{3/6}$	1

If $X = 2$: $f_{Y|X=2}(y) = \frac{f(2,y)}{f_X(2)} = \frac{f(2,y)}{12/36}$ $\Longrightarrow$

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_{Y|X=2}(y)$	$\dfrac{f(2,\,1)}{12/36}$ $=\dfrac{2/36}{12/36}$ $=\mathbf{2/12}$	$\dfrac{f(2,\,2)}{12/36}$ $=\dfrac{4/36}{12/36}$ $=\mathbf{4/12}$	$\dfrac{f(2,\,3)}{12/36}$ $=\dfrac{6/36}{12/36}$ $=\mathbf{6/12}$	1

If $X = 3$: $f_{Y|X=3}(y) = \frac{f(3,y)}{f_X(3)} = \frac{f(3,y)}{18/36}$ $\Longrightarrow$

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_{Y|X=2}(y)$	$\dfrac{f(3,\,1)}{18/36}$ $=\dfrac{2/36}{18/36}$ $=\mathbf{3/18}$	$\dfrac{f(3,\,2)}{18/36}$ $=\dfrac{4/36}{18/36}$ $=\mathbf{6/18}$	$\dfrac{f(3,\,3)}{18/36}$ $=\dfrac{6/36}{18/36}$ $=\mathbf{9/18}$	1

Note: Since X and Y are independent so $\boldsymbol{f_{Y|X}(y) = \frac{f(x,y)}{f_X(x)} = \frac{f_X(x)f_Y(y)}{f_X(x)} = f_Y(y) = \frac{y}{6}}$

Q4) Let X and Y be continuous random variables having joint density function

$$f(x,y) = \begin{cases} c(x^2 + y^2) & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

Determine:
  a. The constant c.
  b. $P\left(X < \frac{1}{2}, Y > \frac{1}{2}\right)$
  c. $P\left(\frac{1}{4} < X < \frac{3}{4}\right)$.

d. $P\left(Y < \frac{1}{2}\right)$.

e. Whether X and Y are independent.

Solution :

$$f(x,y) = C(x^2 + y^2) , \qquad 0 < x < 1, \ 0 < y < 1$$

a) We know that $\iint_{(x,y)\in C} f(x,y) \, dx \, dy = 1$

$$\int_0^1 \int_0^1 C\,(x^2 + y^2)dx\,dy = 1 \;\Rightarrow\; C\int_0^1\left[\int_0^1 (x^2 + y^2)\,dx\right]dy = 1$$

$$\Rightarrow\; C\int_0^1\left[\frac{x^3}{3} + xy^2\right]_0^1 dy = 1 \;\Rightarrow\; C\int_0^1\left(\frac{1}{3} + y^2\right) = 1 \;\Rightarrow\; C\left[\frac{1}{3}y + \frac{y^3}{3}\right]_0^1 = 1$$

$$\Rightarrow\; C\left[\frac{1}{3} + \frac{1}{3}\right] = 1 \;\Rightarrow\; \boldsymbol{C = \frac{3}{2}}$$

b)

$$P\left(X < \frac{1}{2}, Y > \frac{1}{2}\right) = \frac{3}{2}\int_{1/2}^1\left[\int_0^{1/2}(x^2 + y^2)dx\right]dy = \frac{3}{2}\int_{1/2}^1\left[\frac{x^3}{3} + xy^2\right]_0^{1/2} dy$$

$$= \frac{3}{2}\int_{1/2}^1\left(\frac{1}{24} + \frac{1}{2}y^2\right)dy = \frac{3}{2}\left[\frac{y}{24} + \frac{y^3}{6}\right]_{\frac{1}{2}}^1 = \frac{3}{2}\left[\left(\frac{1}{24} + \frac{1}{6}\right) - \left(\frac{1}{48} + \frac{1}{48}\right)\right] = \frac{1}{4}$$

c)

$$P\left(\frac{1}{4} < X < \frac{3}{4}\right) = \frac{3}{2}\int_0^1\left[\int_{\frac{1}{4}}^{\frac{3}{4}}(x^2 + y^2)dx\right]dy = \frac{3}{2}\int_0^1\left[\frac{x^3}{3} + xy^2\right]_{\frac{1}{4}}^{\frac{3}{4}} dy$$

$$= \frac{3}{2}\int_0^1\left[\left(\frac{9}{64} + \frac{3}{4}y^2\right) - \left(\frac{1}{192} + \frac{1}{4}y^2\right)\right]dy = \frac{3}{2}\int_0^1\left(\frac{13}{96} + \frac{1}{2}y^2\right)dy$$

$$= \frac{3}{2}\left[\frac{13}{96}y + \frac{1}{6}y^3\right]_0^1 = \frac{3}{2}\left(\frac{13}{96} + \frac{1}{6}\right) = \frac{29}{64}$$

d)

$$P\left(Y < \frac{1}{2}\right) = \frac{3}{2}\int_0^{\frac{1}{2}}\left[\int_0^1(x^2 + y^2)dx\right]dy = \frac{3}{2}\int_0^{\frac{1}{2}}\left[\frac{x^3}{3} + xy^2\right]_0^1 dy$$

$$= \frac{3}{2}\int_0^{1/2}\left(\frac{1}{3} + y^2\right)dy = \frac{3}{2}\left[\frac{y}{3} + \frac{y^3}{3}\right]_0^{\frac{1}{2}} = \frac{3}{2}\left[\frac{1}{6} + \frac{1}{24}\right] = \frac{5}{16}$$

e) X and Y are independent if satisfy :
1) $f(x,y) = f_X(x)f_Y(y) \ \forall\, x,y$.
2) the ranges of X and Y are independent.

In this example , we can see that

$$f(x,y) \neq f_X(x)f_Y(y) \quad \therefore X \text{ and } Y \text{ are not independent.}$$

$$f_Y(y) = \frac{3}{2}\int_0^1 (x^2 + y^2)dx = \frac{3}{2}\left[\frac{x^3}{3} + xy^2\right]_0^1 = \frac{3}{2}\left(\frac{1}{3} + y^2\right) = \frac{1}{2} + \frac{3}{2}y^2$$

$$f_X(x) = \frac{3}{2}\int_0^1 (x^2 + y^2)dy = \frac{3}{2}\left[x^2 y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}\left(x^2 + \frac{1}{3}\right) = \frac{3}{2}x^2 + \frac{1}{2}$$

$$f_X(x)f_Y(y) = \left(\frac{3}{2}x^2 + \frac{1}{2}\right)\left(\frac{3}{2}y^2 + \frac{1}{2}\right) = \frac{9}{4}x^2 y^2 + \frac{3}{4}x^2 + \frac{3}{4}y^2 + \frac{1}{4} \neq f(x,y)$$

Q5) For the random variables of **Problem 4**, find the marginal probability function of X and Y.

Solution :
Marginal distribution of X : $f(x) = \int_0^1 f(x,y)dy$

$$f(x) = \frac{3}{2}\int_0^1 (x^2 + y^2)dy = \frac{3}{2}\left[x^2 y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}\left[x^2 + \frac{1}{3}\right] = \frac{3}{2}x^2 + \frac{1}{2}$$

$$\therefore f_X(x) = \frac{3}{2}x^2 + \frac{1}{2} \quad ; 0 < x < 1$$

Marginal distribution of Y : $f(y) = \int_0^1 f(x,y)dx$

$$f(y) = \frac{3}{2}\int_0^1 (x^2 + y^2)dx = \frac{3}{2}\left[\frac{x^3}{3} + y^2 x\right]_0^1 = \frac{3}{2}\left(\frac{1}{3} + y^2\right) = \frac{3}{2}y^2 + \frac{1}{2}$$

$$\therefore f_Y(y) = \frac{3}{2}y^2 + \frac{1}{2} \quad ; \ 0 < y < 1$$

Q6) For the distribution of **Problem 4**, find the conditional probability function of X given Y, Y given X.

Solution :
Conditional distribution $X|Y$:

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{3}{2}(x^2 + y^2)}{\frac{3}{2}y^2 + \frac{1}{2}} = \frac{\frac{3}{2}(x^2 + y^2)}{\frac{3}{2}\left(y^2 + \frac{1}{3}\right)} = \frac{x^2 + y^2}{y^2 + \frac{1}{3}}$$

for $0 < x < 1$ *wher* $0 < y < 1$ fixed value .

Conditional distribution $Y|X$:

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}x^2+\frac{1}{2}} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}\left(x^2+\frac{1}{3}\right)} = \frac{x^2+y^2}{x^2+\frac{1}{3}}$$

for $0 < y < 1$ *wher* $0 < x < 1$ fixed value .

Q7) Let $f(x,y) = \begin{cases} x+y & 0 \le x \le 1 , \ 0 \le y \le 1 \\ 0 & otherwise \end{cases}$

Find the conditional probability function of X given Y, Y given X.

Solution : H.W

$$f(x,y) = x+y \quad ; 0 \le x \le 1 , 0 \le y \le 1$$

Marginal pdf of X : $f(x) = \int_0^1 f(x,y)dy = \int_0^1 (x+y)dy = \left[xy+\frac{y^2}{2}\right]_0^1 = x+\frac{1}{2}$

Marginal pdf of Y : $f(y) = y+\frac{1}{2}$

Conditional distribution $X|Y$:

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{x+y}{y+\frac{1}{2}} = \frac{2(x+y)}{(2y+1)}$$

Conditional distribution $Y|X$:

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}} = \frac{2(x+y)}{(2x+1)}$$

Q8) Let $f(x,y) = \begin{cases} e^{-(x+y)} & x \ge 0 , y \ge 0 \\ 0 & otherwise \end{cases}$

be the joint density function of X and Y. Find the conditional probability function of X given Y, Y given X.

Solution :

Marginal pdf of X : $f(x) = \int_0^\infty e^{-(x+y)}\,dy = e^{-x}[-e^{-y}]_0^\infty = e^{-x}[0+1] = e^{-x}$

Marginal pdf of Y : $f(y) = \int_0^\infty e^{-(x-y)}\,dx = e^{-y}$

Conditional distribution $X|Y$: $f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-(x+y)}}{e^{-y}} = e^{-x}$

Conditional distribution $Y|X$: $f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-(x-y)}}{e^{-x}} = e^{-y}$

Q9) Let X and Y be random variables having joint density function

$$f(x,y) = \begin{cases} c(2x+y) & 0 < x < 1, \quad 0 < y < 2 \\ 0 & otherwise \end{cases}$$

Find:

a. The constant c.

b. $P\left(X > \frac{1}{2}, Y < \frac{3}{2}\right)$.

c. The (marginal) density function of X.

d. The (marginal) density function of Y.

Solution : H.W

Q10) The joint probability function for the random variables X and Y is given in following table, then find:

X \ Y	0	1	2
0	1/18	1/9	1/6
1	1/9	1/18	1/9
2	1/6	1/6	1/18

a. The marginal probability functions of X and Y.

b. $P(1 \leq X < 3, Y \geq 1)$.

c. Determine whether X and Y are independent.

Solution : H.W

Q11) Let X and Y be random variables having joint density function
$$f(x,y) = \begin{cases} x + y & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

Find: a. Var(X).   b. Var(Y).   c. σ_X.   d. σ_Y.   e. σ_{XY}.   f. ρ.

Solution :

$f(x) = \int_0^1 f(x,y)dy = \int_0^1 (x+y)dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$

$f(x) = x + \frac{1}{2} \quad ,0 < x < 1$

$f(y) = y + \frac{1}{2} \quad ,0 < y < 1$

a, b)

$$E(X) = \int_0^1 x\, f(x)dx = \int_0^1 x\left(x + \frac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^2 + \frac{x}{2}\right) dx = \left[\frac{x^3}{3} + \frac{1}{4}x^2\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$E(Y) = \int_0^1 y\, f(y)dy = \frac{7}{12}$$

$$E(X^2) = \int_0^1 x^2\, f(x)dx = \int_0^1 x^2\left(x + \frac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^3 + \frac{x^2}{2}\right) dx = \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$$E(Y^2) = \int_0^1 y^2\, f(y)dy = \frac{5}{12}$$

$$var(X) = E(X^2) - [E(X)]^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

$$var(Y) = E(Y^2) - [E(Y)]^2 = \frac{11}{144}$$

c, d)

$$\sigma_X = \sqrt{var(X)} = \sqrt{11/144} = 0.2764$$

$$\sigma_Y = \sqrt{var(Y)} = \sqrt{11/144} = 0.2764$$

e) $\sigma_{XY} = cov(x,y) = E(XY) - E(X)E(Y)$

$$E(XY) = \int_0^1 \int_0^1 xy\,(x+y)dx\,dy = \int_0^1 \left[\frac{x^3}{3}y + \frac{x^2}{2}y^2\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{1}{3}y + \frac{y^2}{2}\right)dy = \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$cov(x,y) = \frac{2}{6} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = -\frac{1}{144}$$

f) $\rho = cor(x,y) = \dfrac{cov(x,y)}{\sqrt{var(x)var(y)}} = \dfrac{-1/144}{\sqrt{\left(\frac{11}{144}\right)\left(\frac{11}{144}\right)}} = \dfrac{-1}{11} = -0.091$

(Weak negative correlation)

Q12) The joint density function is

$$f(x,y) = \begin{cases} e^{-(x+y)} & x \geq 0\,, y \geq 0 \\ 0 & otherwise \end{cases}$$

Find: a. Var(X).　　b. Var(Y).　　c. σ_X.　　d. σ_Y.　　e. σ_{XY}.　　f. ρ.

Solution : H.W

Q13) Find a. The covariance. b. The correlation coefficient of two random variables X and Y. If $E(X) = 2$, $E(Y) = 3$, $E(XY) = 10$, $E(X^2) = 9$, $E(Y^2) = 16$.

Solution :

a) $cov(x,y) = E(XY) - E(X)E(Y) = 10 - (2)(3) = 4$
b) $var(x) = E(x^2) - [E(x)]^2 = 9 - 2^2 = 5$

$$var(y) = E(y^2) - [E(y)]^2 = 16 - 3^2 = 7$$

$\rho = \dfrac{cov(x,y)}{\sqrt{var(x)var(y)}} = \dfrac{4}{\sqrt{(5)(7)}} = \dfrac{4}{\sqrt{35}} = 0.676123$　(Moderate positive correlation)

Q14) The correlation coefficient of two random variables **X** and **Y** is (**-1/4**) while their variances are **3** and **5**. Find the covariance.

Solution : H.W

$$\rho = -\frac{1}{4} \quad ; var(x) = 3\,, var(y) = 5$$

$$\rho = cor(x,y) = \frac{cov(x,y)}{\sqrt{var(x)var(y)}}$$

$$-\frac{1}{4} = \frac{cov(x,y)}{\sqrt{(3)(5)}} \implies -\frac{1}{4}\sqrt{15} = cov(x,y) \implies cov(x,y) = -0.9682$$

(Strong negative correlation)

Q15) The joint probability function of two **discrete** random variables X and Y is given by $f(x,y) = c(2x + y)$, where x and y can assume all integers such that $0 < x < 1,\ 0 < y < 3$, and $f(x,y) = 0$ otherwise. Find:

   a. The value of the constant c.
   b. $P(X = 2, Y = 1)$.
   c. $P(X \geq 1, Y \leq 2)$

Solution : H.W

$$f(x,y) = c(2x + y)\ \ ; x = 0,1\ ;\ \ y = 0,1,2,3$$

a) $\sum_{i=0}^{3} \sum_{j=0}^{1} f(x_i, y_j)\ = 1$

$$6c + 14c = 1 \ \Rightarrow\ \ 20c = 1 \Rightarrow c = \frac{1}{20}$$

x＼y	0	1	$f_Y(y)$
0	0	2c	2c
1	c	3c	4c
2	2c	4c	6c
3	3c	5c	8c
$f_X(x)$	6c	14c	1

b) $P(X = 2, Y = 1) = 0$

c) $P(X \geq 1, Y \leq 2) = P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 2)$

$$= \frac{1}{20}(2 + 3 + 4) = \frac{9}{20}$$

Q16) For the **Problem 15**, find a. E(X). b. E(Y). c. E(XY). d. E(X²). e. E(Y²). f. Var(X). g. Var(Y). h. Cov(X,Y). i. ρ.

Solution :

$$E(X) = \sum_{x=0}^{1} x\, f(x) = \frac{1}{20}[0 * 6 + 1 * 14] = \frac{14}{20} = \frac{7}{10}$$

$$E(X^2) = \sum_{x=0}^{1} x^2\, f(x) = \frac{1}{20}[0 * 6 + 1 * 14] = \frac{14}{20} = \frac{7}{10}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{7}{10} - \left(\frac{7}{10}\right)^2 = \frac{21}{100}$$

$$E(Y) = \sum_{y=0}^{3} y\, f(y) = \frac{1}{20}[0*2 + 1*4 + 2*6 + 3*8] = \frac{40}{20} = 2$$

$$E(Y^2) = \sum_{y=0}^{3} y^2\, f(y) = \frac{1}{20}[0*2 + 1*4 + 4*6 + 9*8] = \frac{100}{20} = 5$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = 5 - (2)^2 = 1$$

$$E(XY) = \sum_{y=0}^{3}\sum_{x=0}^{1} yx\, f(x,y) = \frac{1}{20}(1*1*3 + 1*2*4 + 1*3*5) = \frac{26}{20} = \frac{13}{10}$$

$$\sigma_{XY} = cov(x,y) = E(XY) - E(X)E(Y) = \frac{13}{10} - \left(\frac{7}{10}\right)(2) = -0.1$$

$$\rho = \frac{cov(x,y)}{\sqrt{var(x)var(y)}} = \frac{-0.1}{\sqrt{\frac{21}{100} * 1}} = -0.2182$$
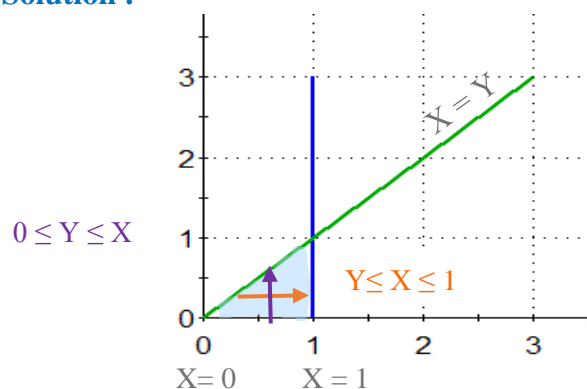
weak negative correlation.

Q17) The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 8xy & 0 \le x \le 1\ ,0 \le y \le x \\ 0 & otherwise \end{cases}$$

Find:

    a. The marginal density of X.
    b. The marginal density of Y.
    c. The conditional density of X.
    d. The conditional density of Y.

Solution :

a) $f_X(x) = \int_0^x f(x,y)dy$

$$= 8\int_0^x xy\, dy = 8x\left[\frac{y^2}{2}\right]_0^x = 8x\left[\frac{x^2}{2}\right] = 4x^3 \quad for\ 0 < x < 1$$

b) $f_Y(y) = \int_y^1 f(x,y)dx$

$$= 8y\int_y^1 x\, dx = 8y\left[\frac{x^2}{2}\right]_y^1 = 4y[1-y^2] = 4y - 4y^3$$

$$f_Y(y) = 4(y - y^3) \quad for\ 0 < y < 1$$

c) Conditional distribution $X|Y$: $f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{8xy}{4(y-y^3)} = \frac{2xy}{y(1-y^2)} = \frac{2x}{1-y^2}$

for $y < x < 1$ wher $0 < y < 1$ fixed value .

d) Conditional distribution $Y|X$: $f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}$

for $0 < y < x$ wher $0 < x < 1$ fixed value .

Q18) Find the conditional expectation of X given Y and Y given X in **Problem 17**.

 Solution :

$$E(X|Y=y) = \int_y^1 x\, f_{X|Y=Y}(x)dx = \frac{2}{1-y^2}\int_y^1 x^2\, dx$$

$$= \frac{2}{1-y^2}\left[\frac{x^3}{3}\right]_y^1 = \frac{2}{3}\frac{1}{1-y^2}(1-y^3) = \frac{2}{3}\frac{(1-y^3)}{(1-y^2)}$$

$$E(Y|X=x) = \int_0^x y\, f_{Y|X=x}(y)dy = \frac{2}{x^2}\int_0^x y^2\, dy = \frac{2}{x^2}\left[\frac{y^3}{3}\right]_0^x = \frac{2}{3}\frac{1}{x^2}[x^3] = \frac{2}{3}x$$

Q19) Find the conditional variance of Y given X for **Problem 17**.

 Solution :
$$var(Y|X) = E(Y^2|X) - E(Y|X)^2$$

$$E(Y^2|X) = \int_0^x y^2 f_{Y|X=x}(y)dy = \frac{2}{x^2}\int_0^x y^3 dy = \frac{2}{x^2}\left[\frac{y^4}{4}\right]_0^x = \frac{2}{4}\frac{1}{x^2}[x^4] = \frac{x^2}{2}$$

$$var(Y|X) = \frac{x^2}{2} - \left[\frac{2}{3}x\right]^2 = \left(\frac{1}{2} - \frac{2^2}{3^2}\right)x^2 = \frac{1}{18}x^2$$

Note: Q13,Q14,Q16,Q18,Q19, Q20, Q21, Q22 $\Rightarrow$ we will solve it later.

Q20) The joint pdf of (X,Y) is given by $f(x, y) = \dfrac{e^{-y}}{y}$; $0 < x < y, \ 0 < y < \infty$.
Find $E(X), E(Y), V(X), V(Y)$ and $Cov(X, Y)$.

   Solution : H.W

$$E(X) = \int_0^\infty \int_0^y x \frac{1}{y} e^{-y} \, dx \, dy = \int_0^\infty \frac{1}{y} e^{-y} \left[\frac{x^2}{2}\right]_0^y dy = \int_0^\infty \frac{1}{2} y \, e^{-y} dy$$

Note: $X \sim$ exponential$(\lambda) \to f(x) = \lambda e^{-\lambda x} \ x > 0$; $E(x) = \frac{1}{\lambda}$; $V(x) = \frac{1}{\lambda^2}$

$$let \ W = e^{-y} \sim Exp(1) \ ; E(W) = V(W) = 1$$

$$\therefore E(X) = \frac{1}{2} E(W) = \frac{1}{2}$$

$$\boldsymbol{or\ use} \int_0^\infty x^a \, e^{-b\,x} \, dx = \frac{\Gamma(a+1)}{b^{a+1}} \ , \qquad \Gamma(a) = (a-1)!$$

$$E(X^2) = \int_0^\infty \int_0^y x^2 \frac{1}{y} e^{-y} \, dx \, dy = \int_0^\infty \frac{1}{y} e^{-y} \left[\frac{x^3}{3}\right]_0^y dy$$

$$= \int_0^\infty \frac{1}{3} y^2 e^{-y} \, dy = \frac{1}{3} E(W^2) = \frac{1}{3}(2) = \frac{2}{3}$$

$$V(X) = \frac{2}{3} - \left(\frac{1}{2}\right)^2 = \frac{5}{12}$$

$$E(Y) = \int_0^\infty \int_0^y y \frac{1}{y} e^{-y} \, dx \, dy = \int_0^\infty e^{-y} \, dy = 1$$

$$E(Y^2) = \int_0^\infty y^2 \, e^{-y} \, dy = 2 \quad ; \quad V(Y) = 2 - 1 = 1$$

$$E(XY) = \int_0^\infty \int_0^y xy \frac{1}{y} e^{-y} \, dx \, dy = \int_0^\infty \int_0^y x e^{-y} \, dx \, dy$$

$$= \int_0^\infty e^{-y} \left[\frac{x^2}{2}\right]_0^y dy = \int_0^\infty \frac{1}{2} y^2 e^{-y} \, dy = \frac{1}{2} \frac{\Gamma(3)}{1^3} = \frac{1}{2}(2!) = 1$$

$$cov(x, y) = E(XY) - E(X)E(Y) = 1 - \frac{1}{2}(1) = \frac{1}{2}$$

Q21) Let (X,Y) have joint density given by

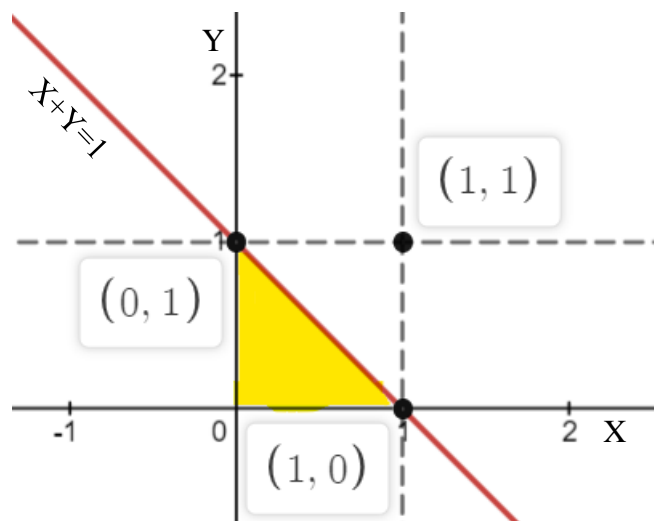$$f(x, y) = 24xy \qquad ; \ 0 < x < 1, \quad 0 < y < 1, \quad x + y < 1$$

Find:

A. The marginal pdf's.
B. The following expectations:

   i. $E(X)$ and $E(X^2)$.

   ii. $E(Y)$ and $E(Y^2)$.

   iii. $E(XY)$ and $E(X^2 Y^3)$.

   iv. $V(X)$, $V(Y)$, $Cov(X,Y)$. Do X and Y have a positive or negative relationship?

Solution : H.W

a) $f(x) = \int_0^{1-x} 24xy \ dy = 24x \left[\frac{y^2}{2} \right]_0^{1-x} = 12\,x(1-x)^2 \ ; \ 0 < x < 1$

      $f(y) = \int_0^{1-y} 24xy \ dx = 24y \left[\frac{x^2}{2} \right]_0^{1-y} = 12\,y(1-y)^2 \ ; \ 0 < y < 1$

b)   Expectations $E(X) = \int_{-\infty}^{\infty} x f(x) \ dx$

$$E(X) = \int_0^1 12x^2(1-x)^2 \ dx = \int_0^1 12x^2(1 - 2x + x^2) \ dx$$

$$= 12 \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right]_0^1 = 12 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right)$$

$$E(X) = \frac{2}{5}$$

Or you can solve it as $E(X) = \int_0^1 \int_0^{1-y} 24\,x^2 y \ dx \ dy = \int_0^1 24y \left[\frac{x^3}{3} \right]_0^{1-y} dy$

$$= \int_0^1 8\,y(1-y)^3 \ dy = \int_0^1 8\,y(1 - 3y + 3y^2 - y^3) \ dy$$

$$= 8 \left[\frac{y^2}{2} - \frac{3y^3}{3} + \frac{3y^4}{4} - \frac{y^5}{5} \right]_0^1 = 8 \left[\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right] = \frac{8}{20} = \frac{2}{5}$$

$$E(X^2) = \int_0^1 12x^3(1-x)^2 \ dx = \int_0^1 12x^3(1-2x+x^2) \ dx = 12\left[\frac{x^4}{4} - \frac{2x^5}{5} + \frac{x^6}{6}\right]_0^1$$

$$E(X^2) = \frac{1}{5}$$

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \frac{1}{5} - \left(\frac{2}{5}\right)^2 = \frac{1}{25} = 0.04$$

By similarly, $E(Y) = \frac{2}{5}$; $E(Y^2) = \frac{1}{5}$; $V(Y) = \frac{1}{25}$

$$E(XY) = \int_0^1 \int_0^1 24 \ x^2 y^2 \ dx \ dy = \int_0^1 24 \ y^2 \left[\frac{x^3}{3}\right]_0^1 dy = \int_0^1 8 \ y^2 \ dy = 8 \left[\frac{y^3}{3}\right]_0^1$$

$$E(XY) = 8$$

$$cov(x,y) = E(XY) - E(X)E(Y) = 8 - \left(\frac{1}{5}\right)^2 = 7.96$$

Since $cov(x,y) > 0$, then X and Y have a positive relationship.

Q22) Let joint pdf of (X,Y) given by $f(x,y) = \frac{1}{y} \ e^{-y} \ e^{-\frac{x}{y}}$; $x > 0, y > 0$, find:
   a. E(X) and E(X²).
   b. E(Y) and E(Y²).
   c. Show that Cov(X,Y)=1.
   d. $\rho(X,Y)$.

Solution :

$$f(x) = \int_0^\infty \frac{1}{y} \ e^{-y} \ e^{-\frac{x}{y}} \ dy = ??$$

$$f(y) = \int_0^\infty \frac{1}{y} \ e^{-y} \ e^{-\frac{x}{y}} \ dx = e^{-y} \left[-e^{-\frac{x}{y}}\right]_0^\infty = e^{-y}(0+1) = e^{-y}$$

$$E(X) = \int_0^\infty \int_0^\infty \frac{x}{y} e^{-y} e^{-\frac{x}{y}} \ dx \ dy = \int_0^\infty \frac{1}{y} e^{-y} \left[\int_0^\infty x e^{-\frac{x}{y}} \ dx\right] dy$$

use $\int_0^\infty x^a \ e^{-b \ x} \ dx = \frac{\Gamma(a+1)}{b^{a+1}}$, $\Gamma(a) = (a-1)!$

$$= \int_0^\infty \frac{1}{y} e^{-y} \ \frac{\Gamma(2)}{\left(\frac{1}{y}\right)^2} \ dy = \int_0^\infty y \ e^{-y} \ dy \qquad \{f(y) = e^{-y} \sim \text{Exponential}(1) , E(y) = 1\}$$

$$E(X) = 1$$

$$E(X^2) = \int_0^\infty \int_0^\infty \frac{x^2}{y} e^{-y} e^{-\frac{x}{y}} \, dx \, dy = \int_0^\infty \frac{1}{y} e^{-y} \left[\int_0^\infty x^2 e^{-\frac{x}{y}} \, dx \right] dy$$

$$= \int_0^\infty \frac{1}{y} e^{-y} \frac{\Gamma(3)}{\left(\frac{1}{y}\right)^3} \, dy = \int_0^\infty 2y^2 \, e^{-y} \, dy = \frac{2\Gamma(3)}{(1)^3} = 4$$

$$E(Y) = \int_0^\infty \int_0^\infty \frac{y}{y} e^{-y} e^{-\frac{x}{y}} \, dx \, dy = \int_0^\infty e^{-y} \left[\int_0^\infty -y \left(-\frac{1}{y}\right) e^{-\frac{x}{y}} \, dx \right] dy$$

$$= \int_0^\infty y \, e^{-y} \left[-e^{-\frac{x}{y}} \right]_0^\infty \, dy = \int_0^\infty y \, e^{-y} \, dy = 1$$

$$E(Y^2) = \int_0^\infty \int_0^\infty \frac{y^2}{y} e^{-y} e^{-\frac{x}{y}} \, dx \, dy = \int_0^\infty y e^{-y} \left[\int_0^\infty -y \left(-\frac{1}{y}\right) e^{-\frac{x}{y}} \, dx \right] dy$$

$$= \int_0^\infty y^2 \, e^{-y} \left[-e^{-\frac{x}{y}} \right]_0^\infty \, dy = \int_0^\infty y^2 \, e^{-y} \, dy = \Gamma(3) = 2$$

$$E(XY) = \int_0^\infty \int_0^\infty \frac{xy}{y} e^{-y} e^{-\frac{x}{y}} \, dx \, dy = \int_0^\infty e^{-y} \int_0^\infty x \, e^{-\frac{x}{y}} \, dx \, dy$$

$$= \int_0^\infty e^{-y} \frac{\Gamma(2)}{\left(\frac{1}{y}\right)^2} \, dy = \int_0^\infty y^2 \, e^{-y} \, dy = \Gamma(3) = 2$$

$$COV(X,Y) = E(XY) - E(X)E(Y) = 2 - (1)(1) = 1$$

Note: