Continuous Random Variables

Q1. If the continuous random variable X has mean μ=16 and variance σ^2=5, then $P(X = 16)$ is

(A) 0.0625 (B) 0.5 (C) 0.0 (D) None of these.

Solution :

$P(X = 16) = 0$ (Because X is Continuous r.v)

Q2. Consider the probability density function:

$$f(x) = \begin{cases} k\sqrt{x}\,, & 0 < x < 1 \\ 0, & elswhere \end{cases}$$

1) The value of k is:

(A)1 (B) 0.5 (C) 1.5 (D) 0.667

2) The probability $P(0.3 < X \leq 0.6)$

(A)0.4647 (B) 0.3004 (C) 0.1643 (D) 0.4500

3) The expected value of X, E(X) is,

(A)0.6 (B) 1.5 (C) 1 (D) 0.667

[Hint: $\int \sqrt{x}\, dx = \frac{x^{\frac{3}{2}}}{3/2} + c$]

Solution :

1) We know $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_0^1 k\sqrt{x}\ dx = 1 \quad \rightarrow \quad k\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = 1 \ \rightarrow \ \frac{2}{3}k = 1 \ \rightarrow \ k = \frac{3}{2} = 1.5$$

$$\therefore\ f(x) = \frac{3}{2}\sqrt{x}\ , \quad 0 < x < 1$$

2) $P(0.3 \leq X \leq 0.6) = \int_{0.3}^{0.6} \frac{3}{2}\sqrt{x}\ dx = \frac{3}{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_{0.3}^{0.6} = (0.6)^{3/2} - (0.3)^{\frac{3}{2}} = 0.3004$

3) $E(x) = \int_{-\infty}^{\infty} xf(x)\, dx = \int_0^1 \frac{3}{2}\, x\sqrt{x}\ dx = \int_0^1 \frac{3}{2}\, x^{\frac{3}{2}}\ dx = \frac{3}{2}\left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 = 0.5999\ \approx 0.6$

Q3. If the cumulative distribution function of the random variable X having the form:

$$P(X \leq x) = F(x) = \begin{cases} 0\,, & x < 0 \\ x/(x+1)\,, & x \geq 0 \end{cases}$$

Then

1) $P(0\ <\ X\ <\ 2)$ equals to

(a) 0.555 (b) 0.333 (c) 0.667 (d) none of these.

2) If $P(\ X \leq\ k)\ =\ 0.5$, then k equals to

(a) 5 (b) 0.5 (c) 1 (d) 1.5

3) The pdf is

(a) $x/(x+1)$ (b) x (c) $1/(x+1)^2$ (d) $x/(x+1)^2$

Solution :

1) $P(0 < X < 2) = F(2) - F(0) = \frac{2}{2+1} - \frac{0}{0+1} = \frac{2}{3} = 0.667$

2) $P(X \leq k) = 0.5 \quad \rightarrow F(k) = 0.5 \quad \rightarrow \frac{k}{k+1} = 0.5 \quad \rightarrow k = 0.5k + 0.5$

$\rightarrow k(1 - 0.5) = 0.5 \quad \rightarrow 0.5k = 0.5 \quad \rightarrow \boldsymbol{k = 1}$

3) $f(x) = \frac{d}{dx} F(x)$

$f(x) = \frac{d}{dx}\left[\frac{x}{x+1}\right] = \frac{(x+1)(1)-x\,(1)}{(x+1)^2} = \frac{1}{(x+1\,)^2} \qquad , x \geq 0$

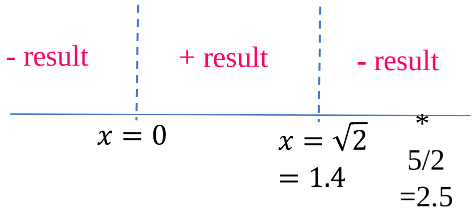
Q4. For each function below, determine if it can be probability density function. If so, determine $\boldsymbol{c}$.

a. $f_1(x) = c(2x - x^3) \quad ; for\ 0 < x < \frac{5}{2}$
b. $f_2(x) = c(2x - x^2) \quad ; for\ 0 < x < \frac{5}{2}$
c. $f_3(x) = c(2x^2 - 4x) \quad ; for\ 0 < x < 3$
d. $f_4(x) = c(2x^2 - 4x) \quad ; for\ 0 < x < 2$

Solution :

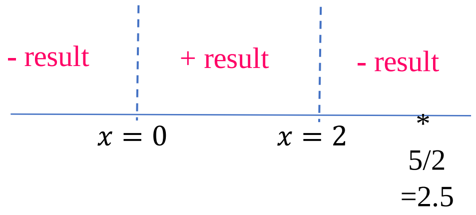
A probability density function must satisfy two requirements:
1) $f(x)$ must be nonnegative for each value of the random variable.
2) the integral over all values of the random variable must equal one.

a) $(2x - x^3) = 0 \ \rightarrow \ x(2 - x^2) = 0 \ \rightarrow \begin{cases} x = 0 \\ or \\ x = \pm\sqrt{2} \end{cases}$



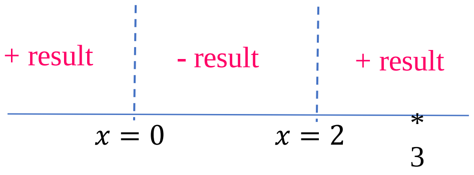
Thus, $(2x - x^3)$ change its sign from (+ result) to (- result) after $x = \sqrt{2}$.
Therefore, $f_1(x)$ is **not** pdf at $0 < x < \frac{5}{2}$.

b) $(2x - x^2) \ \rightarrow x(2 - x) = 0 \quad \rightarrow \begin{cases} x = 0 \\ or \\ x = 2 \end{cases}$



Thus, $f_2(x)$ is **not** pdf at $0 < x < \frac{5}{2}$.

c) $(2x^2-4x) \quad \rightarrow \quad 2x(x-2)=0 \quad \rightarrow \left\{ or \begin{matrix} x=0 \\ x=2 \end{matrix} \right.$



Thus, $f_3(x)$ is **not** pdf at $0<x<3$.

d) $(2x^2-4x) \quad \rightarrow \quad 2x(x-2)=0 \quad \rightarrow \left\{ or \begin{matrix} x=0 \\ x=2 \end{matrix} \right.$

Thus, $f_4(x)$ is pdf at $0<x<2$.

To determine $\boldsymbol{C}$ by use $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$C\int_0^2 (2x^2-4x)\,dx=1 \quad \rightarrow \qquad \mathrm{C}\left[\int_0^2 2x^2\,dx-\int_0^2 4x\,dx\right]=1$$

$$\mathrm{C}\left[\frac{2x^3}{3}-\frac{4x^2}{2}\right]_0^2=1 \;\rightarrow\; C\left[\frac{2}{3}\,(2^3)-2(2^2)\right]=1 \;\rightarrow\; -\frac{8}{3}C\;=1 \;\rightarrow\; \boldsymbol{C}=-\frac{\mathbf{3}}{\mathbf{8}}$$

Q5.The r.v. X has pdf $f(x)=\begin{cases} C(1-x^2)\ , & -1<x<1 \\ \quad 0, & elswhere \end{cases}$

a. What is the value of **C**.

b. Find the following probabilities using the *pdf* of X:
  - i. $P(X<0)$
  - ii. $P(X\geq\frac{1}{2})$
  - iii. $P(-\frac{1}{2}<X\leq\frac{1}{2})$
  - iv. $P(X>1)$

~~c. Graph the pdf f(x). Show $P(X\geq-\frac{1}{2})$ on the graph.~~

d. What is the *cdf* of X.

e. Find the probabilities in (b) using the *cdf.*

Solution :

a) We know $\int_{-\infty}^{\infty} f(x)\,dx=1 \quad \rightarrow \quad \int_{-1}^{1} c(1-x^2)\,dx=1$

$$\rightarrow \quad c\left[x-\frac{x^3}{3}\right]_{-1}^{1}=1 \quad \rightarrow c\left[1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right]=1 \quad \rightarrow c\left[2-\frac{2}{3}\right]=1$$

$$\rightarrow \frac{4}{3}\,c=1 \quad \rightarrow \quad \boldsymbol{c}=\frac{\mathbf{3}}{\mathbf{4}}=\mathbf{0.75}$$

$$\therefore\; f(x)=\frac{3}{4}(1-x^2) \qquad , \qquad -1\leq x\leq 1$$

b) Using pdf

i. $P(x<0)=\frac{3}{4}\int_{-1}^{0}(1-x^2)\,dx=\frac{3}{4}\left[x-\frac{x^3}{3}\right]_{-1}^{0}=\frac{3}{4}\left[0-\left(-1+\frac{1}{3}\right)\right]=\frac{1}{2}$

ii. $P\left(x\geq\frac{1}{2}\right)=\frac{3}{4}\int_{0.5}^{1}(1-x^2)\,dx=\frac{3}{4}\left[x-\frac{x^3}{3}\right]_{0.5}^{1}=\frac{3}{4}\left[\left(1-\frac{1}{3}\right)-\left(\frac{1}{2}-\frac{1}{24}\right)\right]=\frac{5}{32}$

iii. $P\left(-\frac{1}{2}\leq x<\frac{1}{2}\right)=\frac{3}{4}\int_{-0.5}^{0.5}(1-x^2)\,dx=\frac{3}{4}\left[x-\frac{x^3}{3}\right]_{-0.5}^{0.5}$

$=\frac{3}{4}\left[\left(\frac{1}{2}-\frac{1}{24}\right)-\left(-\frac{1}{2}+\frac{1}{24}\right)\right]=\frac{11}{16}$

iv. $P(X>1)=0$

c) *Delete*

d) $$F(x)=\begin{cases}0 & , \quad x<-1\\ \frac{3}{4}\int_{-1}^{x}(1-t^2)\,dt=\frac{3}{4}\left[t-\frac{t^3}{3}\right]_{-1}^{x}=\boldsymbol{\frac{3}{4}x-\frac{1}{4}x^3+\frac{1}{2}} & , \quad -1\leq x<1\\ 1 & , \quad 1\leq x\end{cases}$$

e) Using cdf

i. $P(x<0)=F(0)=\frac{3}{4}(0)-\frac{1}{4}(0)^3+\frac{1}{2}=\frac{1}{2}$

ii. $P\left(x\geq\frac{1}{2}\right)=1-P\left(X<\frac{1}{2}\right)=1-F\left(\frac{1}{2}\right)=1-\left[\frac{3}{4}\left(\frac{1}{2}\right)-\frac{1}{4}\left(\frac{1}{2}\right)^3+\frac{1}{2}\right]$

$=1-\frac{27}{32}=\frac{5}{32}$

iii. $P\left(-\frac{1}{2}\leq x<\frac{1}{2}\right)=F\left(\frac{1}{2}\right)-F\left(-\frac{1}{2}\right)=\frac{27}{32}-\left[\frac{3}{4}\left(\frac{-1}{2}\right)-\frac{1}{4}\left(\frac{-1}{2}\right)^3+\frac{1}{2}\right]=\frac{11}{16}$

iv. $P(X>1)=1-P(X\leq1)=1-F(1)=1-1=0$

Q6.Suppose continuous r.v. X has density function $f(x)=\begin{cases}C\,x^2\ , & 1<x<2\\ 0, & elswhere\end{cases}$

a. Find the value of the constant C.
b. Find $P(X\geq\frac{3}{2})$.
c. Find the cumulative distribution function of X.
d. Find $P(X\geq\frac{3}{2})$ using the cdf.

Solution :

a) We know $\int_{-\infty}^{\infty}f(x)\,dx=1\ \rightarrow\ \int_{1}^{2}cx^2\,dx=1\ \rightarrow \mathrm{C}\left[\frac{x^3}{3}\right]_{1}^{2}=1$

$\rightarrow C\left(\frac{8}{3}-\frac{1}{3}\right)=1\ \rightarrow\ \frac{7}{3}C\ =1\ \rightarrow\ C=\frac{3}{7}\qquad \therefore f(x)=\frac{3}{7}x^2\ ,\ \ 1<x<2$

b) $P\left(X \geq \frac{3}{2}\right) = \frac{3}{7}\int_{3/2}^{2} x^2\,dx = \frac{3}{7}\left[\frac{x^3}{3}\right]_{3/2}^{2} = \frac{1}{7}\left(8 - \frac{27}{8}\right) = \frac{37}{56}$

c) $F(x) = \begin{cases} 0 & , \quad x < 1 \\ \frac{3}{7}\int_{1}^{x} t^2\,dt = \frac{3}{7}\left[\frac{t^3}{3}\right]_{1}^{x} = \frac{\mathbf{1}}{\mathbf{7}}(x^3 - 1) = \frac{x^3}{7} - \frac{1}{7} & , \quad 1 \leq x < 2 \\ 1 & , \quad 2 \leq x \end{cases}$

d) $P\left(X \geq \frac{3}{2}\right) = 1 - p\left(x \leq \frac{3}{2}\right) = 1 - F\left(\frac{3}{2}\right) = 1 - \left(\frac{1}{7}\,\frac{27}{8} - \frac{1}{7}\right) = \frac{37}{56}$

Q7.& Q8.: Deleted

Q9.A system can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = Cxe^{-x/2} \qquad ; x > 0$$

a. What is the probability that the system functions for at least 5 months.
b. What is the probability that the system functions from 3 to 6 months.
c. What is the probability that the system functions less than 1 month.

Solution :

a) To find C, we know that $C\int_0^{\infty} x\,e^{-\frac{x}{2}}\,dx = 1$

$$\left[\, use \int_0^{\infty} x^a\,e^{-b\,x}\,dx = \frac{\Gamma(a+1)}{b^{a+1}}\,, \qquad \Gamma(a) = (a-1)!\,\right]$$

$$C\,\frac{\Gamma(2)}{\left(\frac{1}{2}\right)^2} = 1 \Rightarrow \qquad C\,\frac{1}{1/4} = 1 \;\Rightarrow \qquad 4C = 1 \Rightarrow \qquad \boldsymbol{C = \frac{1}{4}}$$

$$P(X \geq 5) = C\int_5^{\infty} x\,e^{-\frac{x}{2}}\,dx$$

By use calculator

$$P(X \geq 5) = 1 - P(X < 5) = 1 - C\int_0^{5} x\,e^{-\frac{x}{2}}\,dx = 1 - 2.8508\,C = 0.2873$$

Or by use Integration by Parts $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$

$$P(X \geq 5) = \frac{1}{4}\int_5^{\infty} x\,e^{-\frac{x}{2}}\,dx$$

Let $u = x \;\rightarrow\; du = dx$

$dv = e^{-\left(\frac{x}{2}\right)}\,dx \;\rightarrow\; v = -2\,e^{-\frac{x}{2}}$

$$= \frac{1}{4}\left[\left[-2x\,e^{-\frac{x}{2}}\right]_5^{\infty} + 2\int_5^{\infty} e^{-\frac{x}{2}}\,dx\right] = \frac{1}{4}\left[\left(0 + 10e^{-\frac{5}{2}}\right) + 2\,(-2)\int_5^{\infty} -\frac{1}{2}\,e^{-\frac{x}{2}}\,dx\right]$$

$$= \frac{1}{4}\left[10e^{-\frac{5}{2}} - 4\left[e^{-\frac{x}{2}}\right]_5^{\infty}\right] = \frac{1}{4}\left[10e^{-\frac{5}{2}} + 4e^{-\frac{5}{2}}\right] = \frac{1}{4}\left[14e^{-\frac{5}{2}}\right] = 0.2873$$

b) By use calculator $P(3 < X < 6) = \frac{1}{4}\int_3^{6} x\,e^{-\frac{x}{2}}\,dx = 0.3587$

Or by use Integration by Parts $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$

Let $u = x \rightarrow \quad du = dx \quad ; \quad dv = e^{-\left(\frac{x}{2}\right)} dx \rightarrow \quad v = -2\,e^{-\frac{x}{2}}$

$$= \frac{1}{4}\left[\left[-2x\,e^{-\frac{x}{2}}\right]_3^6 + 2\int_3^6 e^{-\frac{x}{2}}\,dx\ \right] = \frac{1}{4}\left[\left(-12e^{-\frac{6}{2}} + 6e^{-\frac{3}{2}}\right) + 2\,(-2)\int_3^6 -\frac{1}{2}\,e^{-\frac{x}{2}}\,dx\right]$$

$$\frac{1}{4}\left[\left(-12e^{-3} + 6e^{-\frac{3}{2}}\right) - 4\left[e^{-\frac{x}{2}}\right]_3^6\right] = \frac{1}{4}\left[-12e^{-3} + 6e^{-\frac{3}{2}} - 4e^{-3} + 4e^{-\frac{3}{2}}\right]$$

$$= \frac{1}{4}\left[10e^{-\frac{3}{2}} - 16e^{-3}\right] = \frac{1}{4}(1.435)$$

c) By use calculator $P(X < 1) = \frac{1}{4}\int_0^1 x\,e^{-\frac{x}{2}}\ dx = 0.0902$

Or by use Integration by Parts $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$

Let $u = x \rightarrow \quad du = dx$

$dv = e^{-\left(\frac{x}{2}\right)} dx \rightarrow \quad v = -2\,e^{-\frac{x}{2}}$

$$= \frac{1}{4}\left[\left[-2x\,e^{-\frac{x}{2}}\right]_0^1 + 2\int_0^1 e^{-\frac{x}{2}}\,dx\ \right] = \frac{1}{4}\left[\left(-2e^{-\frac{1}{2}}\right) + 2\,(-2)\int_0^1 -\frac{1}{2}\,e^{-\frac{x}{2}}\,dx\right]$$

$$= \frac{1}{4}\left[-2e^{-\frac{1}{2}} - 4\left[e^{-\frac{x}{2}}\right]_0^1\right] = \frac{1}{4}\left[-2e^{-\frac{1}{2}} - 4e^{-\frac{1}{2}} + 4\right] = \frac{0.3608}{4} = 0.0902$$

Q10. The cumulative distribution function of a continuous r.v. Y is given by

$$F(y) = \begin{cases} 0\ , & y \le 3 \\ 1 - \dfrac{9}{y^2}\ , & y > 3 \end{cases}$$

Find

a. $P(Y \le 5)$.

b. $P(Y > 8)$.

c. The *pdf* of Y

Solution :

a) $P(Y \le 5) = F(5) = 1 - \frac{9}{25} = \frac{16}{25} = 0.64$

b) $P(Y > 8) = 1 - P(Y \le 8) = 1 - F(8) = 1 - \left(1 - \frac{9}{64}\right) = \frac{9}{64} = 0.1406$

c) $f(y) \underset{\xleftarrow[derivative]{}}{\xrightarrow{integration}} F(y)$

$$f(y) = \frac{d}{dy}F(y) = \ F(y)' \ = \frac{18}{y^3}$$

$$\therefore\ \ f(x) = \begin{cases} \dfrac{18}{y^3}\ , & y > 3 \\ 0\ , & 0.w \end{cases}$$

Q11) If the density function of the continuous r.v. X is $f(x) = \begin{cases} x \;, & 0 < x < 1 \\ 2 - x \;, & 1 \le x < C \\ 0 \;, & 0.w \end{cases}$

Find

a. The value of C.

b. The cumulative distribution function of X.

c. $P(0.8 < X < 0.6\,C)$. ~~Graph f(x) and F(x).~~

Solution :

a) $\int_{-\infty}^{\infty} f(x)\;\, dx \;= 1 \;\rightarrow\; \int_0^1 x\;\, dx \;+\; \int_1^C (2-x)\;\, dx \;= 1$

$$\left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^C = 1 \;\rightarrow \frac{1}{2} + \left(2C - \frac{C^2}{2}\right) - \left(2 - \frac{1}{2}\right) = 1 \;\rightarrow\; \frac{1}{2} + 2C - \frac{C^2}{2} - \frac{3}{2} \;= 1$$

$$\rightarrow\; 2C - \frac{C^2}{2} - 1 \;= 1 \quad\rightarrow\; 2C - \frac{C^2}{2} - 2 = 0 \;\;\rightarrow\; -\frac{C^2}{2} + 2C - 2 = 0$$

by use

$a = -\dfrac{1}{2}\,, b = 2\,, c = -2$

$$C = \frac{-b\; \pm \sqrt{b^2\; - 4ac}}{2a}$$

$$= \frac{-2 \pm\sqrt{4-4}}{2(-\frac{1}{2})} = -\frac{2}{-1} = 2$$

Or

$$-\frac{1}{2}[C^2 - 4C + 4] = 0$$

$$C^2 - 4C + 4 = 0$$

$$(C-2)^2 = 0$$

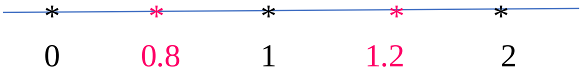
$$C - 2 = 0 \;\rightarrow C = 2$$

$$\therefore f(x) = \begin{cases} x \;, & 0 < x < 1 \\ 2 - x \;, & 1 \le x < 2 \\ 0 \;, & 0.w \end{cases}$$

b) $F(x) = P(X \le x)$

$$F(x) \;\;= \begin{cases} 0 & ,\; \text{x} < 0 \\ \int_0^x x\; dx = \left[\frac{1}{2}x^2\right]_0^x = \boldsymbol{\frac{1}{2}x^2} & ,\; 0 \le x < 1 \\ \int_0^1 x\; dx + \int_1^x (2-x)\; dx = \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x = \boldsymbol{2x - \frac{x^2}{2} - 1} & ,1 \le x < 2 \\ 1 & ,\;\; 2 \le \text{x} \end{cases}$$

c) $P(0.8 < X < 0.6\,C) = P(0.8 < X < 1.2) \quad , wher\; 0.6C = 0.6(2) = 1.2$



By use *pdf*

$$= \int_{0.8}^{1} x\; dx \;+\; \int_1^{1.2} (2-x)dx$$

$$= \left[\frac{x^2}{2}\right]_{0.8}^{1} + \left[2x - \frac{x^2}{2}\right]_1^{1.2} = 0.36$$

By use *cdf*

$$P(X < 1.2) - P(X < 0.8)$$

$$= F(1.2) - F(0.8)$$

$$= \left[2(1.2) - \frac{1.2^2}{2} - 1\right] - \left[\frac{1}{2}(0.8^2)\right]$$

$$= 0.68 - 0.32 = 0.36$$

Summary

- $P(X \geq a) = P(X > a)$ [This NOT true for discrete r.v]
- $f(y) \underset{derivative}{\overset{integration}{\rightleftarrows}} F(y) \Rightarrow \begin{cases} f(x) = \frac{d}{dx} F(x) \\ F(x) = \int_{-\infty}^{x} f(x)\, dx \end{cases}$
- Compute probabilities using cdf:

  $P(a < X < b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$
- $\int_a^b e^{c\,x}\, dx = \left[\frac{1}{c} e^{c\,x}\right]_a^b , c \in \mathbb{R}$
- $e^{-\infty} = 0 ;\quad e^{0} = 1;\quad e^{\infty} = \infty$
- $Y = \frac{u}{v} , v \neq 0 \quad \Rightarrow \frac{dY}{dx} = \frac{v\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$
- $\int_0^{\infty} x^a\, e^{-b\,x}\, dx = \frac{\Gamma(a+1)}{b^{a+1}}$; $a, b \in \mathbb{R}$;

$\Gamma(a) = (a-1)!$; $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$; $\Gamma(a+1) = a\,\Gamma(a) \to \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\sqrt{\pi}$

NOTE:
How to find Roots of equation by Calculator ?

MODE >>> 5:EQN >>> choose equation >>>

You can watch: https://www.youtube.com/watch?v=l_b-gYSlxgc