Markov Chain

Q.1:

Assume that the probability of rain tomorrow is 0.5 if it is raining today, and assume that the probability of its being clear (no rain) tomorrow is 0.9 if it is clear today. Also assume that these probabilities do not change if information is also provided about the weather before today.

a) Explain why the stated assumptions imply that the Markovian property holds for the evolution of the weather.

b) Formulate the evolution of the weather as a Markov chain by defining its states and giving its (one-step) transition matrix.

c) Find the steady-state probabilities (π_0, π_1).

Answers:

P(Rain tomorrow | Rain today) = 0.5   and   P(Clear tomorrow | Clear today) = 0.9

Then,

P(Clear tomorrow | Rain today) = 1- P(Rain tomorrow | Rain today) = 1– 0.5= 0.5

P(Rain tomorrow | Clear today)= 1- P(Clear tomorrow | Clear today)= 1 – 0.9 = 0.1

a) Since the probability of the rain tomorrow is only dependent on the weather today.

b) Defining the states as:

0 = prob. of rain.

1 = prob. of no rain.

$$P = \begin{array}{c} State \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix} \end{array}$$

c) $\pi = \pi P$

$$\pi = [\pi_0 \quad \pi_1] \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix}$$

$$\pi_0 = 0.5\pi_0 + 0.1\pi_1 \qquad (1)$$

$$\pi_1 = 0.5\pi_0 + 0.9\pi_1 \qquad (2)$$

$$\pi_0 + \pi_1 = 1 \qquad (3)$$

From (1) we get:

$0.5\pi_0 = 0.1\pi_1 \Rightarrow \pi_0 = 0.2\pi_1$ (4)

From (2) we get:

$0.1\pi_1 = 0.5\pi_0 \Rightarrow \pi_1 = 5\pi_0$ (5)

Substitute (4) in (3) we get:

$$0.2\pi_1 + \pi_1 = 1 \Rightarrow 1.2\pi_1 = 1 \Rightarrow \pi_1 = \frac{5}{6} = 0.8333$$

$\boxed{\pi_1 = 0.8333}$

Substitute π_1 in (4) we get:

$$\pi_0 = 0.2\pi_1 = 0.2\left(\frac{5}{6}\right) \Rightarrow \pi_0 = \frac{1}{6} = 0.1667$$

$\boxed{\pi_0 = 0.1667}$

$$\pi = (\pi_0, \pi_1) = (0.8333, 0.1667)$$

Q.2:

Consider the inventory example. Dave's Photography Store stocks certain model cameras that can be ordered weekly. Let $D_1, D_2, \ldots$ represent the *demand* for this camera during the first week, second week, …, respectively. Let X_0 represent the number of cameras *on hand* at the outset. Let $X_1, X_2, \ldots$ represent the number of cameras on hand at the end of week 1, week 2, …, respectively. Assume that $X_0 = 2$, so that week 1 begins with two cameras on hand.

As the owner of the store, Dave would like to learn more inventory level at the end of each week, X_t, while using the current ordering policy described below. At the end of each week t (Saturday night), the store places an order that is delivered in time for the next opening of the store on Monday. The store uses the following order policy:

If $\mathrm{X_t} = 0$, order 2 cameras.
If $\mathrm{X_t} > 0$, do not order any cameras.

The demand, D_t, now has the following probability distribution:

$P\{D = 0\} = 0.25, P\{D = 1\} = 0.50, P\{D \geq 2\} = 0.25.$

(a) Construct the (one-step) transition matrix.
(b) Find the steady-state probabilities of the state of this Markov chain.
(c) Assuming that the store pays a storage cost for each camera remaining on the shelf at the end of the week according to the function $C(0) = 0, C(1) =$ \$2, and $C(2) =$ \$8, find the long-run expected average storage cost per week.

Answers:

$\mathrm{D_t}$ = number of cameras sold in week t if the inventory is not depleted (not empty).
$\mathrm{X_t}$ = number of cameras on hand (available in stock) at the end of week t.
States $\mathrm{i, j} = 0,1,2$.
Possible values for $\mathrm{X_t}$:

$$\mathrm{X_{t+1}} = \begin{cases} \max\{2 - \mathrm{D_{t+1}}, 0\} & \text{if } \mathrm{X_t} = 0 \\ \max\{\mathrm{X_t} - \mathrm{D_{t+1}}, 0\} & \text{if } \mathrm{X_t} \geq 1 \text{ } or \text{ } \mathrm{X_t} = 1,2 \end{cases}$$

for $\mathrm{t} = 0,1,2, \ldots$

(a) The one-step matrix is

$$P = \begin{array}{c} State \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} P(D_{t+1} \geq 2) & P(D_{t+1} = 1) & P(D_{t+1} = 0) \\ P(D_{t+1} \geq 1) & P(D_{t+1} = 0) & 0 \\ P(D_{t+1} \geq 2) & P(D_{t+1} = 1) & P(D_{t+1} = 0) \end{bmatrix} \end{array}$$

A transition from $\mathrm{X_t} = 0 \text{ } to \text{ } \mathrm{X_{t+1}} = 0$ implies that the demand for cameras in week t+1 is 2 or more, after 2 cameras added to the depleted inventory at the beginning of the week.

Since $X_{t+1} \leq X_t \text{ } then \text{ } p_{12} = 0$

$P_{00} = P(X_{t+1} = 0|X_t = 0) = P(D_{t+1} \geq 2)$
$P_{01} = P(X_{t+1} = 1|X_t = 0) = P(D_{t+1} = 1)$
$P_{02} = P(X_{t+1} = 2|X_t = 0) = P(D_{t+1} = 0)$

$P_{10} = P(X_{t+1} = 0|X_t = 1) = P(D_{t+1} \geq 1)$
$P_{11} = P(X_{t+1} = 1|X_t = 1) = P(D_{t+1} = 0)$
$P_{12} = P(X_{t+1} = 2|X_t = 1) = 0$

$P_{20} = P(X_{t+1} = 0|X_t = 2) = P(D_{t+1} \geq 2)$
$P_{21} = P(X_{t+1} = 1|X_t = 2) = P(D_{t+1} = 1)$
$P_{22} = P(X_{t+1} = 2|X_t = 2) = P(D_{t+1} = 0)$

$$\begin{array}{ccc} & State & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ P = & \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.75 & 0.25 & 0 \\ 0.25 & 0.5 & 0.25 \end{bmatrix} \end{array}$$

(b) The steady-states are $\pi = (\pi_0, \pi_1, \pi_2)$

$$\pi = \pi P \Rightarrow [\pi_0 \quad \pi_1 \quad \pi_2] \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.75 & 0.25 & 0 \\ 0.25 & 0.5 & 0.25 \end{bmatrix} =$$

$$\pi_0 = 0.25\pi_0 + 0.75\pi_1 + 0.25\pi_2 \qquad (1)$$

$$\pi_1 = 0.5\pi_0 + 0.25\pi_1 + 0.5\pi_2 \qquad (2)$$

$$\pi_2 = 0.25\pi_0 + 0.25\pi_2 \qquad (3)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad (4)$$

Rewrite the equations as:

$$0.75\pi_0 = 0.75\pi_1 + 0.25\pi_2 \qquad (1)$$

$$0.75\pi_1 = 0.5\pi_0 + 0.5\pi_2 \qquad (2)$$

$$0.75\pi_2 = 0.25\pi_0 \qquad (3)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad (4)$$

A method of solving four equations with three variables is by eliminating one of the variables $\pi_0, \pi_1, or\ \pi_2$.

From (1) and (2):

$$0.75\pi_0 = 0.75\pi_1 + 0.25\pi_2$$

$$0.75\pi_1 = 0.5\pi_0 + 0.5\pi_2$$

Multiply (1) by -2:

$-1.5\pi_0 = -1.5\pi_1$ ~~$- 0.5\pi_2$~~

$0.75\pi_1 = 0.5\pi_0$ ~~$+ 0.5\pi_2$~~

Switch between π_0 and π_1:

$$-1.5\pi_0 = -1.5\pi_1$$

$$-0.5\pi_0 = -0.75\pi_1$$

Add both equations:

$$-2\pi_0 = -2.25\pi_1 \Rightarrow \pi_0 = 1.125\pi_1 \qquad (5)$$

Substitute (5) in (3):

$$0.75\pi_2 = 0.25(1.125\pi_1)$$

$$0.75\pi_2 = 0.281\pi_1 \Rightarrow \pi_2 = 0.375\pi_1 \qquad (6)$$

Substitute (5) and (6) in (4):

$$1.125\pi_1 + \pi_1 + 0.375\pi_1 = 1 \Rightarrow 2.5\,\pi_1 = 1 \Rightarrow \boxed{\pi_1 = 0.4}$$

Substitute $\pi_1 = 0.4$ in (5) and (6):

$$\pi_0 = 1.125(0.4) \Rightarrow \boxed{\pi_0 = 0.45}$$

$$\pi_2 = 0.375(0.4) \Rightarrow \boxed{\pi_2 = 0.15}$$

$\pi = (\pi_0, \pi_1, \pi_2) = (0.45, 0.4, 0.15)$.

(c) The long-run expected average storage cost per week:

$$E(C) = \pi * C$$
$$= 0\,\pi_0 + 2\,\pi_1 + 8\,\pi_2$$
$$= 0(0.45) + 2(0.4) + 8(0.15) = \$2\ /\ week$$

Q.3:

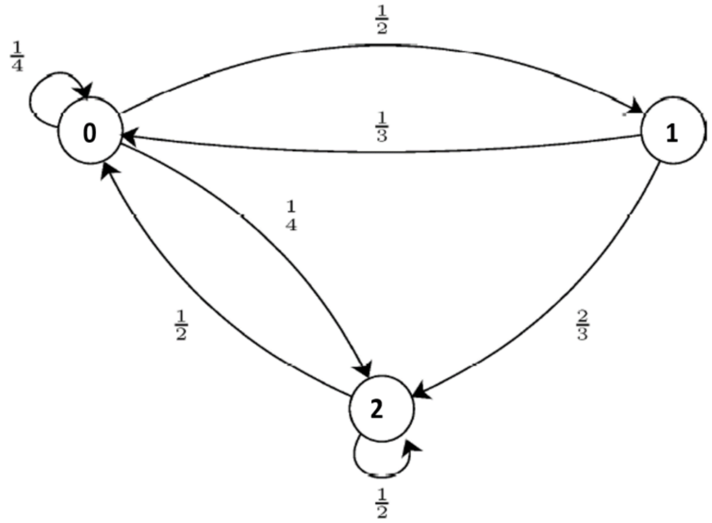
Consider a Markov chain with three possible states, and the following transition probabilities:

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

a) Draw the state transition diagram for the above Markov chain.
b) Find $P(X_4 = 2 | X_3 = 1)$.
c) Find $P(X_3 = 1 | X_2 = 1)$.
d) Find the steady-state probabilities (π_0, π_1, π_2).

Answers:

a) The state transition diagram is



b) $P(X_4 = 2|X_3 = 1) = p_{12} = \frac{2}{3}$

c) $P(X_3 = 1|X_2 = 1) = p_{11} = 0$

d) $\pi = \pi P \Rightarrow [\pi_0 \quad \pi_1 \quad \pi_2]\begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix} =$

$$\pi_0 = \frac{1}{4}\pi_0 + \frac{1}{3}\pi_1 + \frac{1}{2}\pi_2 \qquad (1)$$

$$\pi_1 = \frac{1}{2}\pi_0 \qquad (2)$$

$$\pi_2 = \frac{1}{4}\pi_0 + \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 \qquad (3)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad (4)$$

Rewrite the equations as:

$$\frac{3}{4}\pi_0 = \frac{1}{3}\pi_1 + \frac{1}{2}\pi_2 \qquad (1)$$

$$\pi_1 = \frac{1}{2}\pi_0 \qquad (2)$$

$$\frac{1}{2}\pi_2 = \frac{1}{4}\pi_0 + \frac{2}{3}\pi_1 \qquad (3)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad (4)$$

Use (2) in (3)

$$\frac{1}{2}\pi_2 = \frac{1}{4}\pi_0 + \frac{2}{3}\pi_1$$

$$\frac{1}{2}\pi_2 = \frac{1}{4}\pi_0 + \frac{2}{3}(\frac{1}{2}\pi_0)$$

$$\frac{1}{2}\pi_2 = \frac{7}{12}\pi_0$$

$$\pi_2 = \frac{7}{6}\pi_0 \qquad (5)$$

Use (2) and (5) in (4)

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\pi_0 + \frac{1}{2}\pi_0 + \frac{7}{6}\pi_0 = 1$$

$$\frac{8}{3}\pi_0 = 1$$

$$\pi_0 = \frac{3}{8} \qquad (6)$$

Use (6) in (2) and (5)

$$\pi_1 = \frac{1}{2}\pi_0 \qquad (2)$$

$$\pi_1 = \frac{1}{2}\left(\frac{3}{8}\right) = \frac{3}{16}$$

$$\pi_2 = \frac{7}{6}\pi_0 \qquad (5)$$

$$\pi_2 = \frac{7}{6}\left(\frac{3}{8}\right) = \frac{7}{16}$$

$$\pi = (\pi_0, \pi_1, \pi_2) = \left(\frac{3}{8}, \frac{3}{16}, \frac{7}{16}\right).$$

Queueing Theory

Terminology:

a) ***N(t)***: State of the system at time *t* (number of customers in the queueing system which includes customers in service)

b) $\boldsymbol{P_n}$: probability that exactly *n* customers are in the queueing system.

c) $\boldsymbol{S}$: number of servers in the queueing system.

d) $\boldsymbol{L}$: expected number of customers in the system.

e) $\boldsymbol{L_q}$: expected number of customers in the queue

f) $\boldsymbol{\omega}$: waiting time in the system (includes service time) for each customer

g) $\boldsymbol{W = E(\omega)}$: expected time in the system

h) $\boldsymbol{\omega_q}$: waiting time in the queue (exclude service time)

i) $\boldsymbol{W_q = E(\omega_q)}$: expected time in the queue

Steady-state Equations:

λ: mean arrival rate (expected number of arrivals per unit time)

μ: mean service rate (expected number of customers completing service per unit time)

$\rho = \frac{\lambda}{\mu}$ is the utilization factor

So, $P_0 = 1 - \rho$ and $P_n = \rho^n P_0$

1) $L = \frac{\lambda}{\mu - \lambda}$

2) $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$

3) $W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda}$

4) $W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu - \lambda)}$

5) $P(\omega > t) = e^{-\mu(1-\rho)t}$

6) $P(\omega_q > t) = \rho \times e^{-\mu(1-\rho)t}$

Q.1:

Consider a telephone booth in public where a person can make a paid call. The arrival to the telephone booth is considered Poisson process, and the interarrival and service times are Exponentially distributed. The average arrival rate is 3.75 user every 0.5 hour, and the average length of the phone call is 4 minutes.

1. Probability that the phone will be in use.
2. Expected number of units in the queue.
3. Expected waiting time in the queue.
4. Expected number of units in the system.
5. Expected waiting time in the system
6. What is the probability that an arrival will have to wait in queue for service?
7. What is the probability that zero units in system?
8. What is the probability that exactly 3 units in system?
9. What is the probability that an arrival will not have to wait in queue for service?
10. What is the probability that there are 3 or more units in the system?
11. What is the probability that an arrival will have to wait more than 6 min in queue?
12. What is the probability that more than 5 units in system?
13. What is the probability that an arrival will have to wait more than 8 min in system?

Answers:

Average arrival rate: $\frac{3.75}{\lambda} = \frac{1}{2}$ hour $\rightarrow$ $\boldsymbol{\lambda = 7.5}$ **user / hour**

Average service rate: $\frac{1}{\mu} = 4$ min $\rightarrow$ $\mu = 0.25$ user / min $\rightarrow$ $\boldsymbol{\mu = 15}$ **user / hour**

1. Probability that the phone will be in use.

$$\rho = \frac{\lambda}{\mu} = \frac{7.5}{15} = 0.5$$

2. Expected number of units in the queue.

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{7.5^2}{15(15 - 7.5)}$$

$$L_q = 0.5\,(units)\ person$$

3. Expected waiting time in the queue.

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{0.5}{7.5} = 0.066\ hrs$$

4. Expected number of units in the system.

$$L = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{7.5}{15 - 7.5} = 1\ unit(person)$$

5. Expected waiting time in the system.

$$W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{15 - 7.5} = 0.133 \text{ hour}$$

6. What is the probability that an arrival will have to wait in queue for service?

$$P_{ro} = 1 - P_o$$

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 1 - \left(1 - \frac{\lambda}{\mu}\right)$$

$$P_{ro} = \frac{\lambda}{\mu} = \frac{7.5}{15} = 0.5$$

7. What is the probability that zero units in system?

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 1 - 0.5 = 0.5$$

8. What is the probability that exactly 3 units in system?

$$P_n = P_o \left(\frac{\lambda}{\mu}\right)^n$$

$$P_3 = 0.5(0.5)^3 = 0.0625$$

9. What is the probability that an arrival will not have to wait in queue for service?

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 0.5$$

10. What is the probability that there are 3 or more units in the system?

$$P(\#\ persons \geq 3\ in\ the\ system) = 1 - P(\#\ person \leq 2)$$

$$= 1 - [P_0 + P_1 + P_2]$$

since $P_0 = 1 - \rho\ \ and\ p_n = \rho^n\ P_0$

$$= 1 - [P_0 + P_0\rho + P_0\rho^2]$$

$$= 1 - (1 - \rho)[1 + \rho + \rho^2] = 1 - 1 - \rho - \rho^2 + \rho + \rho^2 + \rho^3 = \rho^3$$

Then, $P_{n\ or\ more} = \left(\frac{\lambda}{\mu}\right)^n \quad = 0.5^3 = 0.125$

11. What is the probability that an arrival will have to wait more than 6 minutes in queue?

[Always watch the unit transformation, here from *minute* to *hour*]

$$P(w_q > t) = \rho\, e^{-\mu(1-\rho)t} = \left(\frac{\lambda}{\mu}\right) e^{(\lambda-\mu)t}$$

$$t = 6\, minutes = \frac{6}{60}\, hours$$

$$P(w_q > t) = (0.5)\, e^{(7.5-15)\left(\frac{6}{60}\right)} = 0.2362$$

12. What is the probability that more than 5 units in system?

$$P(\#\, persons \geq 6\ in\ the\ system) = 1 - P(\#\, person \leq 5) = \rho^6$$

$$P_{n\ or\ more} = \left(\frac{\lambda}{\mu}\right)^n = 0.5^6 = 0.0156$$

13. What is the probability that an arrival will have to wait more than 8 min in **system**?

[Always watch the unit transformation, here from *minute* to *hour*]

$$t = 8\, minutes = \frac{8}{60}\, hours$$

$$P(w > t) = e^{-\mu(1-\rho)t} = e^{(\lambda-\mu)t}$$

$$= e^{(7.5-15)\left(\frac{8}{60}\right)} = 0.3679$$