Ex 8.1

① $f(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}) \\ 0, & x \in [\frac{1}{2}, 1] \end{cases}$

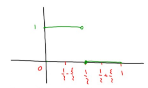
Show that f is integrable on $[0,1]$ and compute $\int_0^1 f(x)dx$

Solution. Let $0 < \varepsilon < 1$.

$$P_\varepsilon = \left\{0, \tfrac{1}{2} - \tfrac{\varepsilon}{2}, \tfrac{1}{2}, \tfrac{1}{2} + \tfrac{\varepsilon}{2}, 1\right\}$$

$[0, \frac{1}{2} - \frac{\varepsilon}{2}]$, $[\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{1}{2} + \frac{\varepsilon}{2}]$, $[\frac{1}{2} + \frac{\varepsilon}{2}, 1]$

$\Delta x_1 = \frac{1}{2} - \frac{\varepsilon}{2}$	$\Delta x_2 = \frac{\varepsilon}{2}$	$\Delta x_3 = \frac{\varepsilon}{2}$	$\Delta x_4 = \frac{1}{2} - \frac{\varepsilon}{2}$
$M_1 = 1$	$M_2 = 1$	$M_3 = 0$	$M_4 = 0$
$m_1 = 1$	$m_2 = 0$	$m_3 = 0$	$m_4 = 0$



$$L(f, P_\varepsilon) = m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 + m_4 \Delta x_4$$
$$= 1 \times \left(\tfrac{1}{2} - \tfrac{\varepsilon}{2}\right) + 0 + 0 + 0 = \tfrac{1}{2} - \tfrac{\varepsilon}{2}$$
$$U(f, P_\varepsilon) = M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3 + M_4 \Delta x_4$$
$$= 1 \times \left(\tfrac{1}{2} - \tfrac{\varepsilon}{2}\right) + 1 \times \tfrac{\varepsilon}{2} + 0 + 0$$
$$= \tfrac{1}{2}$$

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = \tfrac{1}{2} - \left(\tfrac{1}{2} - \tfrac{\varepsilon}{2}\right) = \tfrac{\varepsilon}{2} < \varepsilon$$

By ε-criterion f is Riemann integrable

Computation of $\int_0^1 f(x)dx$.

$$L(f, P_\varepsilon) \le \int_0^1 f(x)dx \le U(f, P_\varepsilon)$$

$\Rightarrow$

$$\tfrac{1}{2} - \tfrac{\varepsilon}{2} \le \int_0^1 f(x)dx \le \tfrac{1}{2}$$

for all $0 < \varepsilon < 1$

We let $\varepsilon \to 0$, so by the squeeze theorem we get $\int_0^1 f(x)dx = \frac{1}{2}$

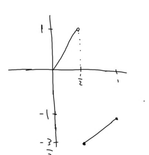
$f(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}) \\ x - 2, & x \in [\frac{1}{2}, 1] \end{cases}$

Solution:

$h = \frac{1}{2n}$, $P_n = \{0, h, 2h, \ldots, 1\}$

$x_k = \frac{k}{2n}$, $k = 0, \ldots, 2n$

$\Big(x_0 = 0$, $x_1 = \frac{1}{2n} = h$, $x_2 = \frac{2}{2n} = 2h$, $\ldots$, $x_n = \frac{1}{2}$, $x_{n+1} = \frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + h$, $\ldots$, $x_{2n} = 1 \Big)$



the function has one point of discontinuity at $x = \frac{1}{2}$, so it is Riemann integrable

$L(f, P_n) = ?$ and $U(f, P_n) = ?$

on $[0, \frac{1}{2})$, $f(x) = 2x$ is increasing

so on $[(k-1)h, kh]$, $M_k = 2kh$, $m_k = 2(k-1)h$, $\Delta x_k = h$

for $k = 1, \ldots, n$

on $[\frac{1}{2}, 1]$, $f(x) = x - 2$, is increasing on $[(k-1)h, kh]$, $k = n+1, \ldots, 2n$

$M_k = kh - 2$, $m_k = (k-1)h - 2$, $\Delta x_k = h$

$$L(f, P_{2n}) = \sum_{k=1}^{2n} m_k \Delta x_k$$
$$= \sum_{k=1}^{n} m_k \Delta x_k + \sum_{k=n+1}^{2n} m_k \Delta x_k$$
$$= \sum_{k=1}^{n} 2(k-1)h(h) + \sum_{k=n+1}^{2n} [(k-1)h - 2](h)$$
$$= 2h^2 \sum_{k=1}^{n} k - 1 + \sum_{k=n+1}^{2n} (k-1)h^2 - \sum_{k=n+1}^{2n} 2h$$
$$= 2h^2 \sum_{k=0}^{n-1} k + h^2 \sum_{k=n}^{2n-1} k - 2h \sum_{k=n+1}^{2n} 1$$
$$= 2h^2 \sum_{k=0}^{n-1} k + h^2 \left(\sum_{k=1}^{2n-1} k - \sum_{k=1}^{n-1} k \right) - 2h(2n - (n+1) + 1)$$
$$= 2h^2 \left(\frac{n(n-1)}{2} \right) + h^2 \left(\frac{(2n-1)(2n)}{2} - \frac{n(n-1)}{2} \right) - 2hn$$
$$= \frac{n(n-1)}{4n^2} + \frac{4n^2 - 2n - n^2 + n}{8n^2} - 1$$
$$= \frac{n-1}{4n} + \frac{3n-1}{8n} - 1$$
$$= \frac{1}{4} - \frac{1}{4n} + \frac{3}{8} - \frac{1}{8n} - 1$$
$$= -\frac{3}{8} - \frac{3}{8n}$$

Similarly

$$U(f, P_{2n}) = \sum_{k=1}^{2n} M_k \Delta x_k$$
$$= \sum_{k=1}^{n} M_k \Delta x_k + \sum_{k=n+1}^{2n} M_k \Delta x_k$$
$$= \sum_{k=1}^{n} 2kh(h) + \sum_{k=n+1}^{2n} (kh - 2)h$$
$$= 2h^2 \sum_{k=1}^{n} k + h^2 \sum_{k=n+1}^{2n} k - 2h \sum_{k=n+1}^{2n} 1$$
$$= \frac{1}{2n^2} \frac{n(n+1)}{2} + \frac{1}{4n^2} \left(\frac{2n(2n+1)}{2} - \frac{n(n+1)}{2} \right) - 1$$
$$= \frac{n+1}{4n} + \frac{4n+2-n-1}{8n} - 1.$$
$$= \frac{1}{4} + \frac{1}{4n} + \frac{3}{8} - \frac{1}{8n} - 1$$
$$= -\frac{3}{8} + \frac{1}{8n}$$

Hence

$$U(f, P_{2n}) - L(f, P_{2n}) = \left(-\frac{3}{8} + \frac{1}{8n}\right) - \left(-\frac{3}{8} - \frac{3}{8n}\right)$$
$$= \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

So by ε-critirion f is integrable

Computation of the integral:

$$\frac{-3}{8} - \frac{3}{8n} \le \int_0^1 f(x)dx \le \frac{-3}{8} + \frac{1}{8n}$$

So by squeeze theorem

$$\int_0^1 f(x)dx = \frac{-3}{8}$$

integrability of $f(x)$ where

$$f(x)=\begin{cases} x, & x\in\mathbb{Q} \\ 0, & x\notin\mathbb{Q}\end{cases}, \quad x\in[0,1]$$

let $P=\{x_0,x_1,\dots,x_n\}$ is a Partition of $[0,1]$

on each interval $[x_{k-1},x_k]$

$M_k=x_k$, $m_k=0$,

$$U(f,P)-L(f,P)=\sum_{k=1}^{n}(M_k-m_k)\Delta x_k$$

$\frac{1}{2}(x_k+x_{k-1})$; x_{k-1} , x_k

$$=\sum_{1}^{n} x_k(x_k-x_{k-1})$$

$$\geq \sum_{k=1}^{n}\frac{1}{2}(x_k+x_{k-1})(x_k-x_{k-1})$$

$$=\frac{1}{2}\sum_{k=1}^{n} x_k^2-x_{k-1}^2$$

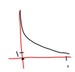
$$=\frac{1}{2}(x_n^2-x_0^2)=\frac{1}{2}$$

for any Partition P $U(f,P)-L(f,P)\geq\frac{1}{2}$

for $\varepsilon<\frac{1}{2}$ there is no Partition s.t.

$U(f,P)-L(f,P)<\varepsilon$, so f is not integrable

$$f(x)=\begin{cases} 1/x, & x\neq 0 \\ 0, & x=0\end{cases}, \quad x\in[0,1]$$



$\lim_{x\to 0^+} f(x)=\infty$. f is unbounded on $[0,1]$

so f is not Riemann integrable

$f(x)=x^2 \quad x\in[0,1]$

P uniform partition of $[0,1]$

$\Delta x_k=\frac{1}{n}$, $x_k=\frac{k}{n}$, $k=0,1,\dots,n$

$x_0=0,\ x_1=\frac{1}{n},\ x_2=\frac{2}{n},\dots,x_k=\frac{k}{n}$

On $[x_{k-1},x_k]$

$$M_k=(x_k^2)=\frac{k^2}{n^2},\quad m_k=\frac{(k-1)^2}{n^2}$$

$$U(f,P_n)=\sum_{k=1}^{n} M_k\Delta x_k=\sum_{k=1}^{n}\frac{(k)^2}{n^2}\frac{1}{n}$$

$$=\frac{1}{n^3}\sum_{k=1}^{n}k^2=\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{(n+1)(2n+1)}{6n^2}$$

$$L(f,P)=\sum_{k=1}^{n} m_k\Delta x_k=\sum_{k=1}^{n}\frac{(k-1)^2}{n^2}\cdot\frac{1}{n}$$

$$=\frac{1}{n^3}\sum_{k=1}^{n}(k-1)^2$$

$$=\frac{1}{n^3}\sum_{k=0}^{n-1}k^2$$

$$=\frac{1}{n^3}\frac{(n-1)(n)(2(n-1)+1)}{6}$$

$$=\frac{(n-1)(2n-1)}{6n^2}$$

$$U(f,P)-L(f,P)=\frac{n(2n+1)}{6n^2}-\frac{(n-1)(2n-1)}{6n^2}$$

$$=\frac{(2n^2+n)-(2n^2-n-2n+1)}{6n^2}$$

$$=\frac{-1+4n}{6n^2}\to 0,\ \text{as } n\to\infty$$

$\Rightarrow f\in R([0,1])$

$$L(f,P_n)\leq\int_0^1 x^2\,dx\leq U(f,P_n)$$

$$\frac{(n-1)(2n-1)}{6n^2}\leq\int_0^1 x^2\,dx\leq\frac{n(2n+1)}{6n^2}$$

By squeeze theorem

$$\int_0^1 f(x)\,dx=\frac{2}{6}=\frac{1}{3}$$

Exercise Suppose: $f,g:[a,b]\to\mathbb{R}$ and $f(x)=g(x)$, for all $x\in[a,b)$

If $f\in R([a,b])$ prove that

$g\in R([a,b])$ and

$$\int_a^b f(x)\,dx=\int_a^b g(x)\,dx.$$

Solution:

f is bounded because $f(x)=g(x)$ for $x\in[a,b)$, g is also bounded

let $M>0$ s.t. $|f(x)|<M$ and $|g(x)|<M$ for all $x\in[a,b]$

$$\sup g(x)\leq\max\Big(\sup_{[a,b]} f(x),\ g(b)\Big)\leq M$$

Let $\varepsilon>0$ and $P=\{x_0,x_1,\dots,x_n\}$ a partition of $[a,b]$ s.t.

$$U(f,P)-L(f,P)<\frac{\varepsilon}{2}$$

$(f\in R[a,b])$

let $\delta>0$ s.t. $\delta<\frac{\varepsilon}{4M}$.

$Q=\{x_0,x_1,\dots,x_{n-1},b-\delta,b=x_n\}$

"Q refine P": $P\subset Q$

we have

$$U(f,Q)\leq U(f,P)$$
$$L(f,P)\leq L(f,Q)$$

$\Rightarrow$

$$U(f,Q)-L(f,Q)\leq U(f,P)-L(f,P)<\frac{\varepsilon}{2}$$

$$\Rightarrow U(g,Q)=\sum_{k=1}^{n-1}M_k^g\Delta x_k+\sup_{[x_{n-1},b-\delta]}g\cdot(b-\delta-x_{n-1})$$

$$+\sup_{[b-\delta,x_n]}g\cdot(\underbrace{x_n-(b-\delta)}_{=\delta})$$

$$\leq U(f,P)+\delta\cdot M$$

$$L(g,Q)=\sum_{k=1}^{n-1}m_k^g\Delta x_k+\inf_{[x_{n-1},b-\delta]}g\cdot(b-\delta-x_{n-1})$$

$$+\inf_{[b-\delta,b]}g\cdot\delta\geq L(f,P)-M\delta$$

So $L(g,Q)\geq L(f,P)-M\delta$

$\Rightarrow$

$$U(g,Q)-L(g,Q)\leq U(f,P)+M\delta-L(f,P)+M\delta\leq U(f,P)-L(f,P)+2M\delta$$

$$\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Then $g\in R([a,b])$

$$\int_a^b f(x)\,dx=\int_a^b g(x)\,dx\ ?$$

$$L(f,Q)\leq\int_a^b f(x)\,dx\leq U(f,Q)$$
$$L(g,Q)\leq\int_a^b g(x)\,dx\leq U(g,Q)$$

$$\int_a^b f(x)\,dx-\int_a^b g(x)\,dx\leq U(f,Q)-L(g,Q)$$

$$\leq U(f,Q)-L(f,Q)+L(f,Q)-L(g,Q)$$

$$\leq\frac{\varepsilon}{2}+M\delta\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Similarly $\int_a^b g(x)\,dx-\int_a^b f(x)\leq\varepsilon$

Therefore

$$\left|\int_a^b f(x)\,dx-\int_a^b g(x)\,dx\right|\leq\varepsilon$$

$\forall\varepsilon>0$, we let $\varepsilon\to 0$, we obtain

$$\int_a^b f(x)\,dx=\int_a^b g(x)\,dx$$

Example

$$f(x)=\begin{cases}\frac{\sin x}{x}, & x\in(0,1] \\ 1 & x=0\end{cases}$$

$$g(x)=\begin{cases}\frac{\sin x}{x}, & x\in(0,1] \\ A, & x=0\end{cases}$$

- f is continuous on $[0,1]\Rightarrow f\in R([0,1])$
- $f(x)=g(x)$, for all $x\in(0,1]$

$\Rightarrow g\in R[0,1]$ and

$$\int_0^1 f(x)\,dx=\int_0^1 g(x)\,dx$$

let f be cont. and non-negative on $[a,b]$, if $\int_a^b f(x)dx = 0$

Prove that $f(x)=0$, $\forall x \in [a,b]$.

Solution:

Proof by contradiction

suppose that $f \not\equiv 0$ then there exists x_0 s.t. $f(x_0) > 0$

take $\varepsilon = \frac{1}{2} f(x_0)$, since f is cont. at x_0, there exists $\delta > 0$ s.t.

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

$$\Rightarrow x \in (x_0 - \delta, x_0 + \delta) \Rightarrow -\varepsilon < f(x) - f(x_0) < \varepsilon$$

$$\Rightarrow f(x) - f(x_0) > -\varepsilon$$

$$\Rightarrow f(x) > f(x_0) - \varepsilon$$

$$\Rightarrow f(x) > \frac{1}{2} f(x_0)$$

$$\Rightarrow \int_a^b f(x)dx \geq \int_{x_0-\delta}^{x_0+\delta} f(x)dx \geq \int_{x_0-\delta}^{x_0+\delta} \frac{1}{2} f(x_0)\,dx$$

$$\geq \frac{1}{2} f(x_0) \int_{x_0-\delta}^{x_0+\delta} 1\,dx$$

$$\geq \delta f(x_0) > 0$$

Contradiction because $\int_a^b f(x)dx = 0$

exercise 3: evaluate.

a) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \cos\left(\frac{k\pi}{n}\right)$

Recall that: $f \in R(a,b)$

$x_k = a + k\frac{b-a}{n}$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)$$

In general $P = \{x_0, \dots, x_n\}$ partition of $[a,b]$,

$\xi = (\xi_1, \dots, \xi_n)$, $\xi_k \in [x_{k-1}, x_k]$

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_\xi = \int_a^b f(x)dx$$

$b-a=1$, $\Delta x_k = \frac{1}{n}$ is uniform Partition

$a=0$, $\Rightarrow \boxed{b=1}$

$x_k = \frac{k}{n}$, $f(x) = \cos(\pi x)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \cos\left(\frac{k\pi}{n}\right) = \int_0^1 \cos(\pi x)dx$$

$$= \frac{1}{\pi}\left[\sin(\pi x)\right]_0^1 = \frac{1}{\pi}(\sin(\pi) - \sin(0\pi))$$

$$= \frac{1}{\pi}(0-0) = 0$$

b) evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}$

we will evaluate the limit by using Riemann sum, so we must identify

- the interval $[a,b]$
- the partition
- the mark
- the function. f

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{1}{1+\left(\frac{k}{n}\right)^2}$$

$\Delta x_k = \frac{1}{n} \Rightarrow b-a=1$

$f(x) = \frac{1}{1+x^2}$, $x_k = \xi_k = \frac{k}{n} \Rightarrow a=0, b=1$

So $$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \int_0^1 \frac{1}{1+x^2}dx$$

$$= \left[\tan^{-1}(x)\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

c) evaluate $\lim_{n\to\infty} \sum_{k=1}^{2^n} \frac{k^3}{2^{4n}}$

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \frac{k^3}{2^{4n}} = \lim_{n\to\infty} \sum_{k=1}^{2^n} \frac{1}{2^n} \cdot \left(\frac{k}{2^n}\right)^3$$

$\Delta x_k = \frac{1}{2^n} \Rightarrow b-a=1$, $\xi_k = x_k = \frac{k}{2^n}$

$\Rightarrow a=0, b=1$

$x_0 = 0$, $x_1 = \frac{1}{2^n}$, $x_2 = \frac{2}{2^n}$, $\dots$, $x_{2^n-1} = 1 - \frac{1}{2^n}$, $x_{2^n} = 1$

$f(x) = x^3$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^3}{2^{4n}} = \int_0^1 x^3 dx = \frac{1}{4}\left[x^4\right]_0^1 = \frac{1}{4}(1-0) = \frac{1}{4}$$

Improper integral exercises:

Ex 8.5.3:

$$f(x)=\begin{cases} 1/\sqrt{x}, & x\in(0,1] \\ 1/x^2, & x>1 \end{cases}$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{\varepsilon\to 0}\left[2\sqrt{x}\,\right|_\varepsilon^1 = \lim_{\varepsilon\to 0}\left[2-2\sqrt{\varepsilon}\right] = 2$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{\varepsilon\to\infty}\int_1^\varepsilon \frac{1}{x^2}\,dx$$

$$= \lim_{\varepsilon\to\infty}\left[\frac{-1}{x}\,\right|_1^\varepsilon = \lim_{\varepsilon\to\infty}\left[\frac{-1}{\varepsilon}+1\right] = 1$$

$$\Rightarrow \int_0^\infty f(x)\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx + \int_1^\infty \frac{1}{x^2}\,dx = 3$$

Ex. 4) Cauchy criterion for improper integral

$\int_a^\infty f(x)\,dx$ converges iff, $\forall \varepsilon>0, \exists N \geq a$

s.t. $b, c > N \Rightarrow \left|\int_b^c f(x)\,dx\right| < \varepsilon$

$\Rightarrow$ let $F(x) = \int_a^x f(t)\,dt$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}\int_a^x f(t)\,dt = \int_a^\infty f(t)\,dt = L$$

let $\varepsilon > 0$ there exists $N > a$ S.T.

$$x > N \Longrightarrow |F(x) - L| < \frac{\varepsilon}{2}$$

then

$b, c > N \Rightarrow |F(b) - L| < \frac{\varepsilon}{2}$

and $|F(c) - L| < \frac{\varepsilon}{2}$

$$\left|\int_b^c f(t)\,dt\right| = \left|\int_b^a f(t)\,dt + \int_a^c f(t)\,dt\right|$$

$$= \left|\int_a^c f(t)\,dt - \int_a^b f(t)\,dt\right|$$

$$= |F(c) - F(b)|$$

$$= |F(c) - L + L - F(b)|$$

$$\leq |F(c) - L| + |F(b) - L|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Ex 8.5.5:

Suppose $f, g \in \mathcal{R}(a,c)$ for all $c > a$

and $|f(x)| \leq g(x)$, $\forall x \in [a,\infty)$

if $\int_a^\infty g(x)\,dx$ is convergent

Prove that $\int_a^\infty f(x)\,dx$ is also convergent.

Solution:

following exercise 4 it suffies

to show that $\forall \varepsilon > 0\ \exists N$ S.T.

$b, c > N \Rightarrow \left|\int_b^c f(x)\,dx\right| < \varepsilon$

(Cauchy critirion for improper integral)

$\int_0^\infty g(x)\,dx$ converges

$\forall \varepsilon > 0\ \exists N$ S.T.

$$\forall b, c > N \Rightarrow \left|\int_b^c g(x)\,dx\right| = \int_b^c g(x)\,dx < \varepsilon$$

$$\left|\int_b^c f(x)\,dx\right| \leq \int_b^c |f(x)|\,dx$$

$$\leq \int_b^c g(x)\,dx < \varepsilon$$

So by Cauchy criterion for improper integral

$\int_0^\infty f(x)\,dx$ converges

application:

$$\int_0^\infty \frac{1}{1+x^4}\,dx$$

- $\frac{1}{1+x^4} < \frac{1}{x^4}$, $\forall x \in (1,\infty)$
- $\int_1^\infty \frac{dx}{x^4}$ conv $\left(\int_1^\infty \frac{dx}{x^\alpha} \text{ conv} \Leftrightarrow \alpha > 1,\ \alpha = 4\right)$

$$\Rightarrow \int_1^\infty \frac{dx}{1+x^4} \text{ cv}$$

$$\int_0^\infty \frac{dx}{1+x^4} = \underbrace{\int_0^1 \frac{1}{1+x^4}\,dx}_{\text{cv}} + \underbrace{\int_1^\infty \frac{dx}{1+x^4}}_{\text{cv}} \text{ cv}$$

$$\boxed{\begin{aligned} &\int_0^1 \frac{dx}{x^\alpha} \text{ cv} \Leftrightarrow \alpha < 1 \\ &\int_1^\infty \frac{dx}{x^\alpha} \text{ cv} \Leftrightarrow \alpha > 1 \end{aligned}}$$

Ex 8) Prove that

$$\int_0^{\infty} \frac{\sin(x)}{x}\,dx \text{ . cv}$$

but $\int_0^{\infty} \frac{|\sin x|}{x}\,dx$ does not

(the integral converges, but not absolutely converges)

By exercise 5) :

If $\int_a^{\infty} |f(x)|\,dx \Rightarrow \int_a^{\infty} f(x)\,dx$ cv

the converse is not true

$$\left(\int_0^{\infty} \frac{\sin(x)}{x}\,dx \text{ cv, but } \int_0^{\infty} \left| \frac{\sin x}{x} \right| dx \text{ . D.}\right.$$

$$\int_1^{\infty} \frac{\sin x}{x}\,dx = \lim_{A \to \infty} \int_1^{A} \frac{\sin x}{x}\,dx$$

integrating by parts,

$$\int_1^{A} \frac{\sin x}{x}\,dx$$

$$\left| \begin{array}{l} u = \frac{1}{x} \\ dv = \sin x\,dx \end{array} \right| \begin{array}{l} du = -\frac{1}{x^2}\,dx \\ v = -\cos(x) \end{array}$$

$$\int_1^{A} \frac{\sin(x)}{x}\,dx = \left.\frac{-\cos(x)}{x}\right|_1^A - \int_1^{A} \frac{\cos(x)}{x^2}\,dx$$

$$= \underbrace{\frac{-\cos(A)}{A}}_{\to 0} + \cos(1) - \int_1^{A} \frac{\cos(x)}{x^2}\,dx$$

. $\left| \frac{\cos x}{x^2} \right| \leq \frac{1}{x^2}$

. $\int_1^{\infty} \frac{dx}{x^2}$ cv

$\Rightarrow \int_1^{\infty} \frac{\cos x}{x}\,dx$ cv

$\Rightarrow \lim_{A \to \infty} \int_1^{A} \frac{\sin x}{x}\,dx = \cos(1) - \int_1^{\infty} \frac{\cos(x)}{x^2}\,dx$

$\Rightarrow \int_1^{\infty} \frac{\sin x}{x}\,dx$ converges

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\int_0^{1} \frac{\sin x}{x}\,dx$ exists

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \int_0^1 + \int_1^{\infty} \frac{\sin x}{x}\,dx \text{ cv}$$

$\int_0^{\infty} \frac{|\sin x|}{x}\,dx$ diverges

$$\int_0^{n\pi} \frac{|\sin(x)|}{x}\,dx = \int_0^{\pi} + \int_{\pi}^{2\pi} + \dots + \int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{x}$$

$$= \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin(x)|}{x}\,dx$$

$\forall k$

$$\int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx \geq \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin(x)|\,dx$$

$t = x - (k-1)\pi$

$dx = dt$

$$\int_{(k-1)\pi}^{k\pi} |\sin(x)|\,dx = \int_0^{\pi} |\sin(t + (k-1)\pi)|\,dt$$

$$= \int_0^{\pi} |\sin(x)|\,dx$$

$\forall k \in \mathbb{N}$

$$\int_{(k-1)\pi}^{k\pi} \frac{|\sin(x)|}{x}\,dx \geq \frac{1}{k\pi} \int_0^{\pi} |\sin(x)|\,dx$$

$$\sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin(x)|}{x}\,dx \geq \left(\frac{1}{\pi}\int_0^{\pi} |\sin(x)|\,dx\right) \sum_{k=1}^{n} \frac{1}{k}$$

$\forall n \in \mathbb{N}$

$$\int_0^{n\pi} \frac{|\sin(x)|}{x}\,dx \geq c \sum_{k=1}^{n} \frac{1}{k} \to \infty$$

where $c = \frac{1}{\pi} \int_0^{\pi} |\sin(x)|\,dx$

$\Rightarrow \lim_{n \to \infty} \int_0^{n\pi} \frac{|\sin(x)|}{x}\,dx = \infty$, the integral div