- Exercises 6 -

Q1: $f(x) = 4x^3 \quad , \quad 0 < x < 1$

$$F_X(x) = P(X \leq x) = \int_0^x 4t^3\,dt = x^4$$

ⓐ $Y = X^4$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^4 \leq y) \\ &= P(X \leq y^{\frac{1}{4}}) \\ &= F_X(y^{\frac{1}{4}}) \\ &= y \qquad ; \quad 0 < y < 1 \end{aligned}$$

domain of Y:
$0 < x < 1$
$0 < x^4 < 1$
$0 < y < 1$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = 1 \qquad ; \quad 0 < y < 1$$

$y \sim$ uniform $(0,1)$

ⓑ $W = e^X$

$$\begin{aligned} F_W(\omega) &= P(W \leq \omega) \\ &= P(e^X \leq \omega) \\ &= P(X \leq \ln \omega) \\ &= (\ln \omega)^4 \end{aligned}$$

domian of w:
$0 < x < 1$
$e^0 < e^x < e^1$
$1 < \omega < e^1$

$$f_W(\omega) = \frac{dF(\omega)}{d\omega} = 4(\ln \omega)^3 \cdot \frac{1}{\omega} \qquad ; \quad 1 < \omega < e$$

ⓒ $Z = \ln X$

$$\begin{aligned} P(Z \leq z) &= P(\ln X \leq z) \\ &= P(X \leq e^z) \\ &= F_X(e^z) \\ &= e^{4z} \end{aligned}$$

domian of Z:
$0 < x < 1$
$\ln 0 < \ln x < \ln 1$
$z < 0$

$$f_Z(z) = \frac{dF(z)}{dz} = 4e^{4z} \qquad ; \quad z < 0$$

Q_2: $X \sim Unif(0,1)$

$$f(x) = \frac{1}{b-a} = \frac{1}{1-0} = 1 \qquad 0<x<1$$

$$F_X(x) = P(X \leq x) = \int_0^x 1 \, dt = x$$

ⓐ $Y = X^{\frac{1}{4}}$

$$F_Y(y) = P(Y \leq y) = P(X^{\frac{1}{4}} \leq y)$$
$$= P(X \leq y^4)$$
$$= F_X(y^4)$$
$$= y^4$$

domian of Y:
$0<x<1$
$0<x^{\frac{1}{4}}<1$
$0<Y<1$

$$f_Y(y) = \frac{dF(y)}{dy} = 4y^3 \quad ; \ 0<y<1$$

ⓑ $W = e^{-X}$

$$F_W(\omega) = P(W \leq \omega) = P(e^{-X} \leq \omega)$$
$$= P(-X \leq \ln \omega)$$
$$= 1 - P(X < -\ln \omega)$$
$$= 1 + \ln \omega$$

domian of W:
$0<x<1$
$0>-x>-1$
$e^0 > e^{-x} > e^{-1}$
$1 > e^{-x} > e^{-1}$
$1 > W > e^{-1}$

$$f_W(\omega) = \frac{dF(\omega)}{d\omega} = \frac{1}{\omega} \quad ; \ e^{-1} < \omega < 1$$

ⓒ $Z = 1 - e^{-X}$

$$F_Z(z) = P(Z \leq z) = P(1 - e^{-X} \leq z)$$
$$= P(-e^{-X} \leq z-1)$$
$$= P(e^{-X} \geq 1-z)$$
$$= P(X \leq -\ln(1-z))$$
$$= -\ln(1-z)$$

$$f(z) = \frac{dF(z)}{dz} = -\frac{1}{1-z}\cdot(-1) = \frac{1}{1-z} \quad ; \quad 0 < Z < 1-e^{-1}$$

$0 < x < 1$
$0 > -x > -1$
$1 > e^{-x} > e^{-1}$
$-1 < -e^{-x} < -e^{-1}$
$0 < 1-e^{-x} < 1-e^{-1}$
$0 < Z < 1-e^{-1}$

Q_3: $f(x) = \theta x^{\theta-1}, \quad 0 < x < 1 \quad ; \quad 0 < \theta < \infty$

$Y = -2\theta \ln X$

- The CDF of X:

$$F(x) = P(X \le x) = \int_0^x \theta t^{\theta-1}\, dt = \frac{\cancel{\theta} t^\theta}{\cancel{\theta}}\Big|_0^x = x^\theta$$

- CDF of $Y = -2\theta \ln x$:

$$F_Y(y) = P(Y \le y) = P(-2\theta \ln x \le y)$$
$$= P\left(\ln x \ge \frac{-y}{2\theta}\right)$$
$$= P\left(X \ge e^{-\frac{y}{2\theta}}\right)$$
$$= 1 - P\left(X < e^{-\frac{y}{2\theta}}\right)$$
$$= 1 - F_X\left(e^{-\frac{y}{2\theta}}\right) \quad \text{(Using cdf of x)}$$
$$= 1 - e^{-\frac{y}{2}}$$

domain of Y:
$0 < x < 1$
$\ln 0 < \ln x < \ln 1$
$\ln x < 0$
$-2\theta \ln x > 0$
$y > 0$

- The pdf of Y:

$$f_Y(y) = \frac{dF(y)}{dy} = \frac{1}{2} e^{-\frac{y}{2}} \quad ; \quad y > 0$$

$\therefore$ Y is an exponential distribution with mean 2

Q_4: $f(x) = \dfrac{e^{-x}}{(1+e^{-x})^2}$; $-\infty < x < \infty$

- The CDF of x:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} \frac{e^{-t}}{(1+e^{-t})^2}\, dt$$

let $u = 1+e^{-t} \Rightarrow du = -e^{-t}\, dt$

$$= \int_{-\infty}^{x} -\frac{1}{u^2}\, du$$

$$= \frac{1}{u}\Big|_{x=-\infty}^{x}$$

$$= \frac{1}{1+e^{-t}}\Big|_{-\infty}^{x}$$

$$= \frac{1}{1+e^{-x}}$$

$\frac{1}{\infty} = 0$ / $e^{-\infty} = 0$ / $e^{\infty} = \infty$

- CDF of Y:

$$F_Y(y) = P(Y \leq y) = P\left(\frac{1}{1+e^{-x}} \leq y\right)$$

$$= P\left(1+e^{-x} \geq \frac{1}{y}\right)$$

$$= P\left(e^{-x} \geq \frac{1}{y} - 1\right)$$

$$= P\left(X \leq -\ln\left(\frac{1}{y}-1\right)\right)$$

$$= F_X\left(-\ln\left(\frac{1}{y}-1\right)\right)$$

$$= \frac{1}{1+\frac{1}{y}-1} = y$$

- The pdf of Y:

$$f_Y(y) = \frac{dF(y)}{dy} = 1 \quad ; \quad 0 < y < 1$$

$\therefore Y \sim Unif(0,1)$

domian of Y:
$-\infty < x < \infty$
$\infty > -x > -\infty$
$\infty > e^{-x} > 0$
$\infty > 1+e^{-x} > 1$
$0 < \frac{1}{1+e^{-x}} < 1$

$e^{-\infty} = 0$
$e^{\infty} = \infty$
$\frac{1}{\infty} = 0$