3.8 Exercises

1. Determine the matrix C given by the following expression

$$C = 2A - 3B,$$

if the matrices A and B are

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 4 \end{pmatrix}.$$

2. Find the product AB and BA for the matrices of the Problem 1.

3. Show that the product AB of the following rectangular matrices is a singular matrix.

$$A = \begin{pmatrix} 6 & -3 \\ 1 & 4 \\ -2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 & -2 \\ 3 & -4 & -1 \end{pmatrix}.$$

4. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 2 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

(a) Compute AB and BA and show that $AB \neq BA$.
(b) Find $(A + B) + C$ and $A + (B + C)$.
(c) Show that $(AB)^T = B^T A^T$.

5. Find a value of x and y such that $AB^T = C^T$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 0 \\ 2 & 1 & 3 \end{pmatrix}, \quad B = [1 \ x \ 1], \quad C = [-2 \ -2 \ y].$$

6. Find the values of a and b such that each of the following matrix is symmetric:

$$\textbf{(a)} \quad A = \begin{pmatrix} 1 & 3 & 5 \\ a+2 & 5 & 6 \\ b+1 & 6 & 7 \end{pmatrix}, \quad \textbf{(b)} \quad B = \begin{pmatrix} -2 & a+b & 2 \\ 3 & 4 & 2a+b \\ 2 & 5 & -3 \end{pmatrix},$$

$$\textbf{(c)} \quad C = \begin{pmatrix} 1 & 4 & a-b \\ 4 & 2 & a+3b \\ 7 & 3 & 4 \end{pmatrix}, \quad \textbf{(d)} \quad D = \begin{pmatrix} 1 & a-4b & 2 \\ 2 & 8 & 6 \\ 7 & a-7b & 8 \end{pmatrix}.$$

7. Which of the following matrices are skew-symmetric ?
(a)

$$A = \begin{pmatrix} 1 & -5 \\ 5 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix},$$

(b)

$$C = \begin{pmatrix} 1 & 9 \\ -9 & 7 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 6 \\ -6 & 2 \end{pmatrix},$$

(c)

$$E = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 4 \\ 2 & -4 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 3 & -3 & -3 \\ 3 & 3 & -3 \\ 3 & 3 & 3 \end{pmatrix}.$$

8. Compute the determinant of each of the following matrix using cofactor expansion along any row or column:

$$A = \begin{pmatrix} \cos x & \sin x & 1 \\ 0 & 3\cos x & -3\sin x \\ 0 & 2\sin x & 2\cos x \end{pmatrix}, \quad B = \begin{pmatrix} x & y & z \\ 0 & x^2 & y \\ 0 & y^2 & x \end{pmatrix}, \quad C = \begin{pmatrix} 2x & 0 & z \\ 0 & 2y & -z \\ z & -z & 2z \end{pmatrix}.$$

9. Compute the determinant of each of the following matrix using cofactor expansion along any row or column:

$$A = \begin{pmatrix} 3 & 7 & 6 \\ 0 & 3 & 5 \\ 7 & 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 11 & -6 & 4 \\ -16 & 8 & 6 \\ 5 & 7 & 12 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & -8 & 11 \\ 10 & 1 & 4 \\ 7 & 10 & 8 \end{pmatrix}.$$

10. Find all zeros (values of x such that $f(x) = 0$) of polynomial $f(x) = \det(A)$ where

$$A = \begin{pmatrix} x-1 & 3 & 2 \\ 3 & x & 1 \\ 2 & 1 & x-2 \end{pmatrix}.$$

11. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

then show that $(AB)^{-1} = B^{-1}A^{-1}$.

12. Find all zeros (values of x such that $f(x) = 0$) of polynomial $f(x) = \det(A)$ where

$$A = \begin{pmatrix} x & 0 & 1 \\ 2 & 1 & 3 \\ 0 & x & 2 \end{pmatrix}.$$

13. Compute the adjoint of each matrix A, and find the inverse of it, if it exists:

(**a**) $A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$, (**b**) $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 1 & 5 & -8 \end{pmatrix}$, (**c**) $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

14. Find all zeros (values of x such that $f(x) = 0$) of polynomial $f(x) = \det(A)$ where

$$A = \begin{pmatrix} x & -8 & 5 & 2 \\ -3 & x & 2 & 1 \\ 3 & 4 & x & 1 \\ 3 & 6 & -5 & 17 \end{pmatrix}.$$

15. Show that $A(Adj\ A) = (Adj\ A)A = \det(A)\mathbf{I}_3$, if

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{pmatrix}.$$

16. Find the inverse and determinant of the adjoint matrix of each following matrix:

$$A = \begin{pmatrix} 4 & 1 & 5 \\ 5 & 6 & 3 \\ 5 & 4 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 & -2 \\ 2 & 5 & 4 \\ 7 & -3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 4 & 0 \\ 3 & 1 & 1 \end{pmatrix}.$$

17. Find the inverse and determinant of the adjoint matrix of each following matrix:

$$A = \begin{pmatrix} 3 & 2 & 5 \\ 2 & 5 & 4 \\ 5 & 4 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 3 & -2 \\ 3 & 5 & 6 \\ -2 & 6 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{pmatrix}.$$

18. Find inverse of each of the following matrix using determinant:

$$A = \begin{pmatrix} 0 & 1 & 5 \\ 3 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 & -2 \\ -4 & 7 & 5 \\ 5 & -4 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 4 & 2 & -4 \\ 6 & 1 & 4 & -3 \\ 4 & 3 & 1 & 3 \\ 8 & 4 & -3 & 2 \end{pmatrix}.$$

19. Use matrices in Problem 15, solve the following systems using matrix inversion method:

$$(\mathbf{a}) \quad A\mathbf{x} = [1, 1, -3]^T, \quad (\mathbf{b}) \quad B\mathbf{x} = [2, 1, 3]^T, \quad (\mathbf{c}) \quad C\mathbf{x} = [1, 0, 1]^T.$$

20. Solve the following systems using the matrix inversion method:

(a)

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & x_3 & = & 4 \\ 5x_1 & - & 2x_2 & - & x_3 & = & -2 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 9 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 3x_3 & = & 2 \\ 5x_1 & + & 3x_2 & + & x_3 & = & 3 \\ 2x_1 & + & 3x_2 & + & x_3 & = & -1 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} 4x_1 & + & x_2 & - & 3x_3 & = & -1 \\ 3x_1 & + & 2x_2 & - & 6x_3 & = & -2 \\ x_1 & - & 5x_2 & + & 3x_3 & = & -3 \end{array}$$

21. Solve the following systems using the matrix inversion method:

(a)

$$\begin{array}{rcrcrcl} 3x_1 & - & 2x_2 & - & 4x_3 & = & 7 \\ 5x_1 & - & 2x_2 & - & 3x_3 & = & 8 \\ 7x_1 & + & 4x_2 & + & 2x_3 & = & 9 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} -3x_1 & + & 4x_2 & + & 3x_3 & = & 11 \\ 5x_1 & + & 3x_2 & + & x_3 & = & 12 \\ x_1 & + & x_2 & + & 5x_3 & = & 10 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} x_1 & + & 4_2 & - & 8x_3 & = & 7 \\ 2x_1 & + & 7x_2 & - & 5x_3 & = & -5 \\ 3x_1 & - & 6x_2 & + & 6x_3 & = & 4 \end{array}$$

22. Use the simple Gaussian elimination method to show that the following system does not have a solution

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & & & = & 1.5 \\ 2x_1 & - & x_2 & - & x_3 & = & 2 \\ 4x_1 & + & 3x_2 & + & x_3 & = & 0 \end{array}$$

23. Solve the Problem 21 using the simple Gaussian elimination method.

24. Solve the following systems using the simple Gaussian elimination method:

(a)

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & & & = & -2 \\ -x_1 & + & 2x_2 & - & x_3 & = & 5 \\ 4x_1 & - & x_2 & + & 4x_3 & = & 1 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & x_3 & = & 5 \\ 5x_1 & - & 3x_2 & + & 2x_3 & = & 7 \\ 2x_1 & - & x_2 & + & x_3 & = & 3 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 2x_2 & + & 4x_3 & = & 3 \\ 4x_1 & + & 9x_2 & + & 16x_3 & = & 1 \end{array}$$

25. Solve the following systems using the simple Gaussian elimination method

(a)

$$\begin{array}{rcrcrcr} 2x_1 & + & 5x_2 & - & 4x_3 & = & 3 \\ 2x_1 & + & 2x_2 & - & x_3 & = & 1 \\ 3x_1 & + & 2x_2 & - & 3x_3 & = & -5 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} & & 2x_2 & - & x_3 & = & 1 \\ 3x_1 & - & x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 3x_2 & - & 5x_3 & = & 1 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & & & = & 3 \\ -x_1 & & & - & 2x_3 & = & -5 \\ -3x_1 & - & 5x_2 & + & x_3 & = & -4 \end{array}$$

26. For what values of a and b the following linear system has no solution or infinitely many solutions.

(a)

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & x_3 & = & 2 \\ -2x_1 & + & x_2 & + & 3x_3 & = & a \\ 2x_1 & & & - & x_3 & = & b \end{array}$$

(b)

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & x_3 & = & 1 \\ x_1 & - & x_2 & + & 3x_3 & = & a \\ 3x_1 & + & 7x_2 & - & 5x_3 & = & b \end{array}$$

(c)

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 3x_3 & = & 3 \\ 3x_1 & + & x_2 & - & 5x_3 & = & a \\ -5x_1 & - & 5x_2 & + & 21x_3 & = & b \end{array}$$

27. Find the value(s) of α so that each of the following linear system has a non-trivial solution:

(a)

$$\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 3x_1 & + & \alpha x_2 & + & 5x_3 & = & 3 \\ x_1 & + & 7x_2 & + & 3x_3 & = & 2 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 2 \\ x_1 & + & 3x_2 & + & 6x_3 & = & 5 \\ 2x_1 & + & 3x_2 & + & \alpha x_3 & = & 6 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} \alpha x_1 & + & x_2 & + & x_3 & = & 7 \\ x_1 & + & x_2 & - & x_3 & = & 2 \\ x_1 & + & x_2 & + & \alpha x_3 & = & 1 \end{array}$$

28. Find the inverse of each of the following matrix by using simple Gauss elimination method:

$$A = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 2 & 2 \\ 2 & 4 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 3 & 2 \\ 3 & 2 & 2 \\ 2 & 6 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 4 & 3 \end{pmatrix}.$$

29. Find the inverse of each of the following matrix by using simple Gauss elimination method:

$$A = \begin{pmatrix} 3 & 2 & 3 \\ 4 & 2 & 2 \\ 2 & 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 & 2 \\ 3 & 2 & 6 \\ 2 & -6 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 2 & 4 \end{pmatrix}.$$

30. Determine the rank of each of the following matrix:

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ 1 & -5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 & 6 \\ -3 & 6 & 4 \\ 5 & 0 & 9 \end{pmatrix}, \quad C = \begin{pmatrix} 17 & 46 & 7 \\ 20 & 49 & 8 \\ 23 & 52 & 9 \end{pmatrix}.$$

31. Solve Problem 25 using the Gaussian elimination with partial pivoting.

32. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Show that the rank of AB is less than or equal to the rank of A.

33. Solve the following linear systems using the Gaussian elimination with partial and without pivoting

(a)

$$\begin{array}{rcrcl} 1.001x_1 & + & 1.5x_2 & = & 0 \\ 2x_1 & + & 3x_2 & = & 1 \end{array}$$

(b)

$$\begin{array}{rcrcr} x_1 & + & 1.001x_2 & = & 2.001 \\ x_1 & + & x_2 & = & 2 \end{array}$$

(c)

$$\begin{array}{rcrcr} 6.122x_1 & + & 1500.5x_2 & = & 1506.622 \\ 2000x_1 & + & 3x_2 & = & 2003 \end{array}$$

34. The elements of the matrix A, the Hilbert matrix, are defined by

$$a_{ij} = 1/(i + j - 1), \quad \text{for} \quad i, j = 1, 2, \ldots, n$$

Find the solution of the system $A\mathbf{x} = \mathbf{b}$ for $n = 4$ and $\mathbf{b} = [1, 2, 3, 4]^T$, using the Gaussian elimination by partial pivoting.

35. Find the LU decomposition of each matrix A using Doolittle's method, and then solve the systems.

(a)

$$A = \begin{pmatrix} 3 & -2 & 1 & 1 \\ -3 & 7 & 4 & -3 \\ 2 & -5 & 3 & 4 \\ 7 & -3 & 2 & 4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix}.$$

(b)

$$A = \begin{pmatrix} 2 & -4 & 5 & 3 \\ 3 & 5 & -4 & 3 \\ 1 & 6 & 2 & 6 \\ 7 & 2 & 5 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 6 \\ 5 \\ 2 \\ 4 \end{pmatrix}.$$

(c)

$$A = \begin{pmatrix} 2 & 2 & 3 & -2 \\ 10 & 2 & 13 & 11 \\ 2 & 5 & 4 & 6 \\ 1 & -4 & -2 & 7 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 10 \\ 14 \\ 11 \\ 9 \end{pmatrix}.$$

36. For what value(s) of α each of the following matrix A is singular using Doolittle's method.

$$(\mathbf{a})\ A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 3 & -1 \\ \alpha & -2 & 3 \end{pmatrix}, \ (\mathbf{b})\ A = \begin{pmatrix} 1 & 5 & 7 \\ 4 & 4 & \alpha \\ -2 & \alpha & 9 \end{pmatrix}, \quad (\mathbf{c})\ A = \begin{pmatrix} 2 & -4 & \alpha \\ 2 & 4 & 3 \\ 4 & -2 & 5 \end{pmatrix},$$

$$(\mathbf{d})\ A = \begin{pmatrix} 2 & \alpha & 1-\alpha \\ 2 & 5 & -2 \\ 2 & 5 & 4 \end{pmatrix}, \ (\mathbf{e})\ A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 2 & 3 \\ 4 & \alpha-2 & 7 \end{pmatrix}, \ (\mathbf{f})\ A = \begin{pmatrix} 1 & 5 & \alpha \\ 1 & 4 & \alpha-2 \\ 1 & -2 & 8 \end{pmatrix}.$$

37. Find the determinant of each of the following matrix using the LU decomposition by Doolittle's method:

$$(\mathbf{a}) \quad A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ 2 & 1 & -6 \end{pmatrix}, \quad (\mathbf{b}) \quad A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$(\mathbf{c}) \quad A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 3 & 2 \\ 4 & 1 & 4 \end{pmatrix}.$$

38. Use the smallest positive integer to find the unique solution each of the linear system of the problem 38 using LU decomposition by Doolittle's method.
(a) $A\mathbf{x} = [2, 3, 2]^T$ **(b)** $A\mathbf{x} = [5, -6, 2]^T$ **(c)** $A\mathbf{x} = [11, 13, 10]^T$.
(d) $A\mathbf{x} = [-8, 11, 8]^T$ **(e)** $A\mathbf{x} = [32, 23, 12]^T$ **(f)** $A\mathbf{x} = [-11, 43, 22]^T$.

39. Find the determinant of each of the following matrix using the LU decomposition by Crout's method:

$$(\mathbf{a}) \quad A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 2 & 1 \\ 2 & 1 & -4 \end{pmatrix}, \quad (\mathbf{b}) \quad A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix},$$

$$(\mathbf{c}) \quad A = \begin{pmatrix} 4 & 4 & 1 \\ 5 & 4 & 2 \\ 1 & 4 & 4 \end{pmatrix}, \quad (\mathbf{d}) \quad A = \begin{pmatrix} 2 & 4 & 5 \\ 3 & 5 & 3 \\ 4 & 3 & 2 \end{pmatrix}.$$

40. Find $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$ and $\|\mathbf{x}\|_\infty$ for the following vectors.
(a) $[2, -1, -6, 3]^T$ **(b)** $[\sin k, \cos k, 3^k]^T$, for a fixed integer k.

41. Find $\|.\|_1$, $\|.\|_\infty$ and $\|.\|_e$ for the following matrices.

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ 1 & -5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 & 6 \\ -3 & 6 & 4 \\ 5 & 0 & 9 \end{pmatrix},$$

$$C = \begin{pmatrix} 17 & 46 & 7 \\ 20 & 49 & 8 \\ 23 & 52 & 9 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 11 & -5 & 2 \\ 6 & 8 & -11 & 6 \\ -4 & -8 & 10 & 14 \\ 13 & 14 & -12 & 9 \end{pmatrix}.$$

42. Consider the following matrices

$$A = \begin{pmatrix} -11 & 7 & -8 \\ 5 & 9 & 6 \\ 6 & 3 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 2 & 7 \\ -12 & 10 & 8 \\ 3 & -15 & 14 \end{pmatrix},$$

$$C = \begin{pmatrix} 5 & -6 & 4 \\ -7 & 8 & 5 \\ 3 & -9 & 12 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 1 & 3 & 5 & 2 \\ -2 & -3 & 4 & 5 \\ 3 & 4 & -2 & 4 \end{pmatrix}.$$

Find $\|.\|_1$ and $\|.\|_\infty$ for (**a**) A^3, (**b**) $A^2 + B^2 + C^2 + D^2$, (**c**) BC and (**d**) $C^2 + D^2$.

43. Compute the condition numbers of the following matrices relative to $\|.\|_\infty$

$$(\mathbf{a}) \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{5} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}, \quad (\mathbf{b}) \begin{pmatrix} 0.03 & 0.01 & -0.02 \\ 0.15 & 0.51 & -0.11 \\ 1.11 & 2.22 & 3.33 \end{pmatrix}, \quad (\mathbf{c}) \begin{pmatrix} 1.11 & 1.98 & 2.01 \\ 1.01 & 1.05 & 2.05 \\ 0.85 & 0.45 & 1.25 \end{pmatrix}.$$

44. The $n \times n$ Hilbert matrix $H^{(n)}$ defined by

$$H_{ij}^{(n)} = \frac{1}{i+j-1}, \qquad 1 \le i,\ j \le n.$$

Find the l_∞-norm of the 10×10 Hilbert matrix.

45. The following linear systems have $\mathbf{x}$ as the exact solution and $\mathbf{x}^*$ is an approximate solution. Compute $\|\mathbf{x} - \mathbf{x}^*\|_\infty$ and $K(A)\dfrac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}$, where $\mathbf{r} = \mathbf{b} - A\mathbf{x}^*$ is the residual vector.

(a)

$$\begin{aligned} 0.89x_1 + 0.53x_2 &= 0.36 \\ 0.47x_1 + 0.28x_2 &= 0.19 \end{aligned}$$

$$\begin{aligned} \mathbf{x} &= [1, -1]^T \\ \mathbf{x}^* &= [0.702, -0.500]^T \end{aligned}$$

(b)

$$\begin{aligned} 0.986x_1 + 0.579x_2 &= 0.235 \\ 0.409x_1 + 0.237x_2 &= 0.107 \end{aligned}$$

$$\begin{aligned} \mathbf{x} &= [2, -3]^T \\ \mathbf{x}^* &= [2.110, -3.170]^T \end{aligned}$$

(c)

$$\begin{aligned} 1.003x_1 + 58.090x_2 &= 68.12 \\ 5.550x_1 + 321.8x_2 &= 377.3 \end{aligned}$$

$$\begin{aligned} \mathbf{x} &= [10, 1]^T \\ \mathbf{x}^* &= [-10, 1]^T \end{aligned}$$

46. Discuss the ill-conditioning (stability) of the linear system

$$\begin{array}{rcrcl} 1.01x_1 & + & 0.99x_2 & = & 2 \\ 0.99x_1 & + & 1.01x_2 & = & 2 \end{array}$$

If $\mathbf{x}^* = [2, 0]^t$ be an approximate solution of the system, then find the residual vector $\mathbf{r}$ and estimate the relative error.

47. The exact solution of the following linear system

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 1 \\ x_1 & + & 1.01x_2 & = & 2 \end{array}$$

is $\mathbf{x} = [-99, 100]^T$. Change the coefficient matrix slightly to

$$\delta A = \begin{pmatrix} 1 & 1 \\ 1 & 0.99 \end{pmatrix},$$

and consider the linear system

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 1 \\ x_1 & + & 0.99x_2 & = & 2 \end{array}$$

Compute the change solution $\delta\mathbf{x}$ of the system. Is the matrix A ill-conditioned?

48. The exact solution of the following linear system

$$\begin{array}{rcrcl} x_1 & + & 3x_2 & = & 4 \\ 1.0001x_1 & + & 3x_2 & = & 4.0001 \end{array}$$

is $\mathbf{x} = [1, 1]^T$. Change the right-hand vector $\mathbf{b}$ slightly to $\delta\mathbf{b} = [4.0001, 4.0003]^T$ and consider the linear system

$$\begin{array}{rcrcl} x_1 & + & 3x_2 & = & 4.0001 \\ 1.0001x_1 & + & 3x_2 & = & 4.0003 \end{array}$$

Compute the change solution $\delta\mathbf{x}$ of the system. Is the matrix A ill-conditioned?

49. The exact solution of the following linear system

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 3 \\ x_1 & + & 1.0005x_2 & = & 3.0010 \end{array}$$

is $\mathbf{x} = [1, 2]^T$. Change the coefficient matrix and the right-hand vector $\mathbf{b}$ slightly to

$$\delta A = \begin{pmatrix} 1 & 1 \\ 1 & 1.001 \end{pmatrix} \quad \text{and} \quad \delta\mathbf{b} = \begin{pmatrix} 2.99 \\ 3.01 \end{pmatrix},$$

and consider the linear system

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 2.99 \\ x_1 & + & 1.001x_2 & = & 3.01 \end{array}$$

Compute the change solution $\delta\mathbf{x}$ of the system. Is the matrix A ill-conditioned?

50. Find the condition number of the following matrix

$$A_n = \begin{pmatrix} 1 & 1 \\ 1 & 1 - \dfrac{1}{n} \end{pmatrix}.$$

Solve the linear system $A_4\mathbf{x} = [2, 2]^T$ and compute the relative residual.

51. Find the Jacobi iteration matrix and its l_∞-norm for each of the following matrix.

$$(\mathbf{a}) \quad \begin{pmatrix} 11 & -3 & 2 \\ 4 & 10 & 3 \\ -2 & 5 & 9 \end{pmatrix}, \qquad (\mathbf{b}) \quad \begin{pmatrix} 7 & 1 & 1 \\ 3 & 13 & 2 \\ -4 & 3 & 14 \end{pmatrix},$$

$$(\mathbf{c}) \quad \begin{pmatrix} 8 & 1 & -1 & 0 \\ 2 & 13 & -2 & 1 \\ -1 & 3 & 15 & 2 \\ 1 & 4 & 5 & 20 \end{pmatrix}, \qquad (\mathbf{d}) \quad \begin{pmatrix} 7 & 1 & -3 & 1 \\ 1 & 10 & 2 & -3 \\ 1 & -5 & 25 & 4 \\ 1 & 2 & 3 & 17 \end{pmatrix}.$$

52. Find the Gauss-Seidel iteration matrix and its l_∞-norm for each of the following matrix.

$$(\mathbf{a}) \quad \begin{pmatrix} 3 & 0 & 1 \\ 1 & 4 & 0 \\ 0 & 2 & 5 \end{pmatrix}, \qquad (\mathbf{b}) \quad \begin{pmatrix} 5 & 2 & 1 \\ 4 & 9 & 2 \\ 3 & 1 & 6 \end{pmatrix}.$$

53. Solve the following linear systems using the Jacobi method, start with initial approximation $\mathbf{x}^{(0)} = 0$ and iterate until $\|\mathbf{x}^{(k+1)} - x^{(k)}\|_\infty \leq 10^{-5}$ for each system.

(a)

$$\begin{array}{rcrcrcr} 4x_1 & - & x_2 & + & x_3 & = & 7 \\ 4x_1 & - & 8x_2 & + & x_3 & = & -21 \\ -2x_1 & + & x_2 & + & 5x_3 & = & 15 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & + & x_3 & = & 5 \\ 2x_1 & + & 6x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & 4x_3 & = & 6 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & + & x_3 & = & 1 \\ x_1 & + & 7x_2 & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & 20x_3 & = & 7 \end{array}$$

54. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} -5 & 1 & 0 \\ 1 & 5 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix}.$$

Find the Jacobi iteration matrix T_J and show that $\|T_J\| < 1$. Use Jacobi method to find first approximate solution $\mathbf{x}^{(1)}$ of the linear system by using $\mathbf{x}^{(0)} = [0, 0, 0]^T$. Also, compute the error bound $\|\mathbf{x} - \mathbf{x}^{(10)}\|$. Compute the number of steps needed to get the accuracy within 10^{-5}.

55. Consider the following system of equations

$$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & + & x_3 & = & 1 \\ x_1 & + & 7x_2 & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & 20x_3 & = & 7 \end{array}$$

(a) Show that the Jacobi method converges by using $\|T_J\|_\infty < 1$.

(b) If the first approximate solution of the system by Jacobi method is $\mathbf{x}^{(1)} = [0.25, 0.57, 0.35]^T$, starting with $\mathbf{x}^{(0)} = [0, 0, 0]^T$, then compute an error estimate $\|\mathbf{x} - \mathbf{x}^{(20)}\|_\infty$.

56. If

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}.$$

Find the Jacobi iteration matrix T_J. If the first approximate solution of the given linear system by the Jacobi method is $[3/4, 4/3, 5/4]^T$, using $\mathbf{x}^{(\mathbf{0})} = [0, 0, 0]^T$, then estimate the number of iterations necessary to obtain approximations accurate to within 10^{-6}.

57. Rearrange the following system such that convergence of Gauss-seidel method is guaranteed. Then use $x^{(0)} = [0, 0, 0]^T$ to find first approximation by Gauss-Seidel method. Also, compute an error bound $\|\mathbf{x} - \mathbf{x}^{(\mathbf{10})}\|$.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 5 & -1 & 1 \\ 0 & 3 & -1 \end{pmatrix} \quad textrmand \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

58. Consider the following system of equations

$$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & + & x_3 & = & 11 \\ -x_1 & + & 2x_2 & & & = & 3 \\ 2x_1 & + & x_2 & + & 4x_3 & = & 16 \end{array}$$

(a) Show that the Gauss-Seidel method converges by using $\|T_G\|_\infty < 1$.

(b) Compute the second approximation $\mathbf{x}^{(2)}$, starting with $\mathbf{x}^{(0)} = [1, 1, 1]^T$.

(c) Compute an error estimate $\|\mathbf{x} - \mathbf{x}^{(2)}\|_\infty$ for your approximation.

59. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} -5 & 2 & 1 \\ 1 & -10 & 1 \\ 1 & 1 & -4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -3 \\ 27 \\ 4 \end{pmatrix}.$$

Find the Gauss-Seidel iteration matrix T_G and show that $\|T_G\| < 1$. Use Gauss-Seidel method to find second approximate solution $\mathbf{x}^{(2)}$ of the linear system using $\mathbf{x}^{(0)} = [-0.5, -2.5, -1.5]^T$. Also, compute the error bound.

60. Consider linear system $A\mathbf{x} = \mathbf{b}$, where the coefficient matrix is

$$A = \begin{pmatrix} 6 & -1 & 1 \\ 1 & 5 & -1 \\ 1 & 2 & 9 \end{pmatrix}.$$

Show that Gauss-Seidel method converges faster than Jacobi method. If the first approximate solution of the given linear system by the Gauss-Seidel method is $[0.5, 0.5, 0.5]^T$, using $\mathbf{x}^{(\mathbf{0})} = [0, 0, 0]^T$, then estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}.

61. Consider the following system

$$\begin{array}{rcrcrcl} 4x_1 & + & x_2 & - & 2x_3 & = & 4 \\ 2x_1 & + & 9x_2 & - & 3x_3 & = & 3 \\ x_1 & - & 2x_2 & + & 8x_3 & = & 2 \end{array}$$

(a) Find the matrix form of both iterative (Jacobi and Gauss-Seidel) methods.

(b) If $\mathbf{x}^{(k)} = [x_1^{(k)}, x_2^{(k)}, x_3^{(k)}]^T$, then writing the iterative forms of part(a) in the component forms and find the exact solution of the given system.

(c) Find formulas for the error $\mathbf{e}^{(k+1)}$ in the $(n+1)th$ step .

(d) Find the second approximation of the error $e^{(2)}$ using the part (c) if $\mathbf{x}^{(0)} = [0, 0, 0]^T$.

62. Consider the following linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 5 & -1 & 3 \\ 4 & 7 & -2 \\ 6 & -3 & 9 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Show that Gauss-Seidel method converges for the given linear system. If the first approximate solution of the given linear system by the Gauss-Seidel method is $\mathbf{x}^{(\mathbf{1})} = [0.2, 0.17, 0.26]^T$, by using initial approximation $\mathbf{x}^{(\mathbf{0})} = [0, 0, 0]^T$, then compute an upper bound $\|\mathbf{x} - \mathbf{x}^{(2)}\|_\infty$. Also, compute number of steps needed to get accuracy within 10^{-4}.

63. Consider the following system

$$\begin{array}{rcrcrcl} 16x_1 & - & 3x_2 & + & 2x_3 & = & 11 \\ x_1 & + & 15x_2 & - & 3x_3 & = & 12 \\ 5x_1 & - & 3x_2 & + & 14x_3 & = & 13 \end{array}$$

(a) Find the matrix form of both iterative (Jacobi and Gauss-Seidel) methods.

(b) If $\mathbf{x}^{(k)} = [x_1^{(k)}, x_2^{(k)}, x_3^{(k)}]^T$, then writing the iterative forms of part(a) in the component forms and find the exact solution of the given system.

(c) Find formulas for the error $\mathbf{e}^{(k+1)}$ in the $(n+1)th$ step .

(d) Find the second approximation of the error $e^{(2)}$ using the part (c) if $\mathbf{x}^{(0)} = [0, 0, 0]^T$.