[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Mid-term Exam II / MATH-244 (Linear Algebra) / Semester 451

Max. Marks: 25 **Max. Time: 1.5 hrs**

Question 1: [Marks: (2+3) + 3]

a) Let P_4 denote the vector space of all real polynomials in x with degree ≤ 4 under the usual addition and scalar multiplication. Then:

(i) Show that $W = \{a + 2b + (a-b)x + (2a+b)x^3 + (a+b)x^4 \mid a, b \in \mathbb{R}\}$ is a subspace of P_4.

(ii) Find a basis of the above vector space W.

Solution:

(i) (1) $W \neq \phi$: $0 = 0 + 2(0) + (0-0)x + (2(0)+0)x^3 + (0+0)x^4 \in W$. Let $r \in \mathbb{R}, u, v \in W$

(2) $\underbrace{[a + 2b + (a-b)x + (2a+b)x^3 + (a+b)x^4]}_{u} + \underbrace{[c + 2d + (c-d)x + (2c+d)x^3 + (c+d)x^4]}_{v}$

$= (a+c) + 2(b+d) + [(a+c) - (b+d)]x + [2(a+c) + (b+d)]x^3 + [(a+c) + (b+d)]x^4$

(3) $r[a + 2b + (a-b)x + (2a+b)x^3 + (a+b)x^4] = ra + 2rb + (ra - rb)x + (2ra + rb)x^3 + (ra + rb)x^4$

$= a(1 + x + 2x^3 + x^4) + b(2 - x + x^3 + x^4)$

(ii) Clearly, $a + 2b + (a-b)x + (2a+b)x^3 + (a+b)x^4 = au_1 + bu_2$ with $u_1 = 1 + x + 2x^3 + x^4$, $u_2 = 2 - x + x^3 + x^4 \in W$. Hence, $span\{u_1, u_2\} = W$. **[1 + 0.5 marks]**

Moreover, $\{u_1, u_2\}$ is linearly independent because

$\alpha u_1 + \beta u_2 = 0 \Longrightarrow \alpha + 2\beta = 0, \alpha - \beta = 0, 2\alpha + \beta = 0, \alpha + \beta = 0 \Longrightarrow \alpha = \beta = 0$ **[1 mark]**.

Thus, $\{u_1, u_2\}$ is a basis of the above vector space W. **[0.5 mark]**

b) Let $\{u_1, u_2, \dots, u_n\}$ be a basis of vector space E. Then show that every element of E has a unique representation as linear combination of the basic vectors $u_1, u_2, \dots, u_n$.

Solution: Since $\{u_1, u_2, \dots, u_n\}$ be a basis of vector space E, $span\{u_1, u_2, \dots, u_n\} = E$ and so each $x \in E$ has a **[1 mark]** representation as linear combination of the vectors $u_1, u_2, \dots, u_n$; such a representation is unique because **[0.5 mark]**

$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = x = \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_n u_n \Rightarrow (\alpha_1 - \beta_1)u_1 + (\alpha_2 - \beta_2)u_2 \dots + (\alpha_n - \beta_n)u_n = 0$

$\Rightarrow \alpha_j - \beta_j = 0, \forall j = 1, 2, \dots, n \Rightarrow \alpha_j = \beta_j = 0, \forall j = 1, 2, \dots, n.$ **[1.5 marks]**

Question 2: [Marks: 3 + 3 + 2]

Let $B = \{u_1, u_2, u_3\}$ be a basis of a vector space V and $C = \{w_1, w_2, w_3\} \subseteq V$ such that:

$$u_1 + w_2 = w_1 + w_3$$
$$u_2 - w_3 = w_1 + w_2$$
$$u_3 + w_1 = w_2 - 2w_3.$$

Then:

a) Show that the set C is a basis of the vector space V.

b) Construct the transition matrix ${}_CP_B$ from the above basis B to the basis C.

c) Find the transition matrix ${}_BP_C$ by using the matrix ${}_CP_B$.

$[v]_C = {}_CP_B [v]_B$

Solution:

a) The above given equations can be written as follows: $u_1 = w_1 - w_2 + w_3$, $u_2 = w_1 + w_2 + w_3$ and $u_3 = -w_1 + w_2 - 2w_3$. So, $C = \{w_1, w_2, w_3\}$ generates V. However, $dim\ V = 3$. Therefore, C is a basis of V. **[1 + 1 + 1 marks]**

b) ${}_CP_B = [[u_1]_C \ [u_2]_C \ \dots \ [u_n]_C] = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$. **[1 +2 marks]**

c) $|{}_CP_B| = -2$, $adj({}_CP_B) = \begin{bmatrix} -3 & 1 & 2 \\ -1 & -1 & 0 \\ -2 & 0 & 2 \end{bmatrix}$. Hence. ${}_BP_C = ({}_CP_B)^{-1} = \frac{1}{|{}_CP_B|} adj({}_CP_B) = \frac{-1}{2}\begin{bmatrix} -3 & 1 & 2 \\ -1 & -1 & 0 \\ -2 & 0 & 2 \end{bmatrix}$. **[0.5 + 1.5 +1 marks]**

Question 3: [Marks: 3 + (1.5+1.5) + 3]

a) Let V be a real inner product space and $u, v, w \in V$ satisfying $u + v = -w$, $||u|| = 3, ||v|| = 5$ and $||w|| = 7$. Then find the angle between the vectors u and v.

b) Show that the set $F = \{(0,1,-1), (1,1,1), (2,-1,-1)\}$ is orthogonal in the Euclidean space $\mathbb{R}^3$. Deduce further that the orthogonal set F is a basis of $\mathbb{R}^3$.

c) Let $G = \{v_1 = (1,1,-1,1), v_2 = (1,1,1,1), v_3 = (-1,1,1,1)\}$ be a basis of vector subspace E of the Euclidean space $\mathbb{R}^4$. Find an orthonormal basis of E.

Solution:

a) $-w = u + v \Rightarrow ||w||^2 = ||-w||^2 = ||u+v||^2 = \langle u+v, u+v \rangle = ||u||^2 + ||v||^2 + 2\langle u, v \rangle$. So that: **[1 mark]**

$\langle u, v \rangle = \frac{1}{2}(||w||^2 - ||u||^2 - ||v||^2) = \frac{1}{2}(7^2 - 3^2 - 5^2) = \frac{15}{2}$. Hence, $\theta = cos^{-1}\frac{\langle u,v \rangle}{||u||||v||} = cos^{-1}\frac{\frac{15}{2}}{(3)(5)} = \frac{\pi}{3}$. **[2 marks]**

b) Since $\langle (0,1,-1), (1,1,1) \rangle = \langle (0,1,-1), (2,-1,-1) \rangle = \langle (1,1,1), (2,-1,-1) \rangle = 0$, F is orthogonal in $\mathbb{R}^3$. **[0.5 mark]**

Hence, F being orthogonal set of (three) non-zero vectors is linearly independent in $\mathbb{R}^3$. **[1 mark]**

Recall that $\dim(\mathbb{R}^3) = 3$. Therefore, the set F is a basis of $\mathbb{R}^3$. **[0.5 + 1 marks]**

c) By applying the Gram-Schmidt orthonormalization algorithm on the basis G for vector subspace E of the Euclidean space $\mathbb{R}^4$, we obtain the following orthonormal basis for E:

$$\{\tfrac{1}{2}(1,1,-1,1),\ \tfrac{1}{2\sqrt{3}}(1,1,3,1),\ \tfrac{1}{2\sqrt{6}}(-4,2,0,2)\}.$$ **[0.5 + 1 + 1.5 marks]**

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$$u_1 = v_1 = (1,1,-1,1)$$

$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2} u_1 = (1,1,1,1) - \frac{2}{4}(1,1,-1,1) = \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right)$$

$$u_3 = v_3 - \frac{\langle v_3, u_1\rangle}{\|u_1\|^2} u_1 - \frac{\langle v_3, u_2\rangle}{\|u_2\|^2} u_2$$

$$= (-1,1,1,1) - 0\cdot u_1 - \frac{2}{3}\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right) = \left(\frac{-4}{3}, \frac{2}{3}, 0, \frac{2}{3}\right)$$

$$\Longrightarrow w_1 = \frac{u_1}{\|u_1\|} = \frac{1}{2}(1,1,-1,1)$$

$$w_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{3}}\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right) = \frac{1}{2\sqrt{3}}(1,1,3,1)$$

$$w_3 = \frac{u_3}{\|u_3\|} = \frac{3}{\sqrt{24}}\left(-\frac{4}{3}, \frac{2}{3}, 0, \frac{2}{3}\right) = \frac{1}{\sqrt{24}}(-4,2,0,2)$$