Solution Key: Mid-term Exam I of MATH-244 (Linear Algebra) / Semester 451

Question 1: [Marks: 4+2+3]**:**

a) Find the reduced row echelon form of the matrix $A=\begin{bmatrix}1 & 0 & 3 & 0\\ -1 & 2 & -3 & -1\\ 0 & -2 & 0 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}$ and use it to find non-trivial solutions of the linear system $AX = O$, where $O = [0\ \ 0\ \ 0\ \ 0]^{\mathbf{T}}$.

Solution: $A = \begin{bmatrix}1 & 0 & 3 & 0\\ -1 & 2 & -3 & -1\\ 0 & -2 & 0 & 0\\ 0 & 0 & 0 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3 & 0\\ 0 & 2 & 0 & -1\\ 0 & -2 & 0 & 0\\ 0 & 0 & 0 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3 & 0\\ 0 & 2 & 0 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$ (REF). [2 marks]

Hence, $(-3t, 0, t, 0), \forall\, 0 \neq t \in \mathbb{R}$, is a non-trivial solution of the system $AX = O$. [2 marks]

b) Let B be a 3×3 matrix with $det(B) = 2$. Compute $det(B^{-1} + adj(B))$.

Solution: $det(B^{-1} + adj(B)) = det(B^{-1} + det(B)\, B^{-1}) = det(B^{-1} + 2B^{-1}) = det(3B^{-1}) = \frac{3^3}{det(B)} = \frac{27}{2}$. [2 marks]

c) Let $P = \begin{bmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 1 & 3 & -1\end{bmatrix}$. Compute $adj(P)$ and use it to find P^{-1}.

Solution: $adj(P) = c^{\mathrm{T}} = \begin{bmatrix}-5 & 2 & 1\\ 1 & -1 & -2\\ 1 & -1 & 1\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}-5 & 1 & 1\\ 2 & -1 & -1\\ 1 & -2 & 1\end{bmatrix}$. $det(P) = \begin{vmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 1 & 3 & -1\end{vmatrix} = -3$. [1.5+.5 marks]

Hence, $P^{-1} = \frac{1}{\det(P)}\, adj(P) = \frac{1}{-3}\begin{bmatrix}-5 & 1 & 1\\ 2 & -1 & -1\\ 1 & -2 & 1\end{bmatrix}$. [1 mark]

Question 2: [Marks: 2+3+3]**:**

a) Give example of an invertible matrix A with $\operatorname{tr}(A) = 0$.

Solution: For any non-zero real number x, the matrix $\boldsymbol{A} = \begin{bmatrix}-x & x\\ x & x\end{bmatrix}$ is invertible because [1 mark]

$|A| = -2x^2 \neq 0$. However, $\operatorname{tr}(A) = 0$. [.5+.5 mark]

b) Find the values of λ for which the matrix $C = \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & \lambda\\ 1 & -1 & 3-2\lambda\end{bmatrix}$ is not invertible.

Solution: Since $det(C) = 0$ for any real value of λ, the matrix C is non-invertible for all $\lambda \in \mathbb{R}$. [1+2 marks]

c) Solve the matrix equation $AZ = X + Y$ for Z, where A is an invertible matrix of size 3,

$X = \begin{bmatrix}-1\\ 2\\ 3\end{bmatrix}$, $Y = \begin{bmatrix}5\\ 0\\ -4\end{bmatrix}$, $AX = \frac{1}{3}X$ and $AY = \frac{1}{2}Y$.

Solution: $AX = \frac{1}{3}X$ gives $A^{-1}X = 3X = \begin{bmatrix}-3\\ 6\\ 9\end{bmatrix}$; similarly, $AY = \frac{1}{2}Y$ gives $A^{-1}Y = \begin{bmatrix}10\\ 0\\ -8\end{bmatrix}$. [2 marks]

Hence, $AZ = X + Y$ implies $Z = A^{-1}X + A^{-1}Y = \begin{bmatrix}-3\\ 6\\ 9\end{bmatrix} + \begin{bmatrix}10\\ 0\\ -8\end{bmatrix} = \begin{bmatrix}7\\ 6\\ 1\end{bmatrix}$. [1 mark]

Question 3: [Marks: 4+4]

a) Find the values of δ for which the following linear system of equations

$$\begin{aligned} x + y + z + t &= 4\\ x + \delta y + z + t &= 4\\ x + y + \delta z + (3-\delta)\,t &= 6\\ 2x + 2y + 2z + (\delta - 5)\,t &= 6 \end{aligned}$$

has: (i) no solution (ii) infinitely many solutions.

Solution: Augmented matrix of the given system $\begin{bmatrix} 1 & 1 & 1 & 1 & :4 \\ 1 & \delta & 1 & 1 & :4 \\ 1 & 1 & \delta & 3-\delta & :6 \\ 2 & 2 & 2 & \delta-5 & :6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 4 \\ 0 & \delta-1 & 0 & 0 & : & 0 \\ 0 & 0 & \delta-1 & -5 & : & 0 \\ 0 & 0 & 0 & \delta-7 & : & -2 \end{bmatrix}$. [2 marks]

Hence, the system has no solution if $\delta = 7, 1$ and no infinitely many solutions . [1+1 marks]

b) Use Cramer's rule to solve the following linear system of equations:

$$\begin{aligned} x + y \quad &= 1 \\ x + 2y + z &= -1 \\ x + 3y - z &= 2. \end{aligned}$$

Solution: Let A denote the matrix of coefficients. Then, $|A| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{vmatrix} = -3$. So, the Cramer's is applicable on the given system. Therefore, [1 mark]

$$x = \frac{\begin{vmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{vmatrix}}{|A|} = \frac{-4}{-3} = \frac{4}{3};\ y = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}}{|A|} = \frac{1}{-3} = \frac{-1}{3} \text{ and } z = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{vmatrix}}{|A|} = \frac{5}{-3} = \frac{-5}{3}.$$ [1+1+1 marks]

$-5-(-1) = -4$

$-1-(-2) = 1$

$7\cancel{4}3+1 = 5$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & :4 \\ 1 & \delta & 1 & 1 & :4 \\ 1 & 1 & \delta & 3-\delta & :6 \\ 2 & 2 & 2 & \delta-5 & :6 \end{bmatrix} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & \delta-1 & 0 & 0 & 0 \\ 0 & 0 & \delta-1 & 2-\delta & 2 \\ 0 & 0 & 0 & \delta-7 & -2 \end{array}\right]$$