

EX1.1: Let X be normal random variable with distribution $N(\theta, 1)$. Let $X_1, X_2, \dots, X_{16}$ be 16 copies of X . Test the hypothesis $H_0: \theta \leq 1$ vs $H_1: \theta > 1$ by γ_{UMP} with size $\alpha_{UMP} = 0.05$

Solution

$$f(x; \theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2} \quad -\infty < x < \infty$$

$f(x; \theta)$ Belongs to the class of exponential family

$$f(x; \theta) = e^{-\frac{1}{2} \log 2\pi - \frac{1}{2}(x-\theta)^2}$$

$$a(\theta) = -\frac{1}{2}\theta^2 \quad b(x) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}x^2 \quad c(\theta) = \theta \quad d(x) = x$$

Since $c(\theta)$ is increasing function of θ then γ_{UMP} reject H_0 if $\sum d(x_i) = \sum x_i > k$ or $\bar{X} > C$

$$0.05 = P(\bar{X} > C | \theta = 1) = P(\bar{X} > \frac{C-1}{\frac{1}{4}}) = P(Z > 4(C-1))$$

Thus

$$4(C-1) = 1.645 \rightarrow C = 1.41125 \text{ and } K = 22.58$$

EX1.2: Let X be gamma random variable with distribution $\text{Gamma}(5, \theta)$. Let $X_1, X_2, \dots, X_6$ be 6 copies of X . Test the hypothesis $H_0: \theta \geq \frac{1}{2}$ vs $H_1: \theta < \frac{1}{2}$ by γ_{UMP} with size $\alpha = 0.05$

Solution

$$f(x; \theta) = \frac{\theta^5}{\Gamma(5)} x^{5-1} e^{-\theta x} \quad 0 < x < \infty$$

$f(x; \theta)$ belongs to the class of exponential family

$$f(x; \theta) = e^{5 \log \theta - \log \Gamma(5) + 4 \log x - \theta x}$$

$$a(\theta) = 5 \log \theta \quad b(x) = 4 \log x - \log \Gamma(5) \quad c(\theta) = -\theta \quad d(x) = x$$

Since $c(\theta)$ is decreasing function of θ then γ_{MP} reject H_0 if $S = \sum d(x_i) = \sum x_i > k$

$$X_i \sim \text{Gamma}(5, \theta) \quad \text{then} \quad \sum X_i \sim \text{Gamma}(5 \times 6, \theta) \quad \text{and} \quad U = 2\theta S \sim \chi^2_{2(5 \times 6)}$$

To find k

$$0.05 = P\left(S > k \mid \theta = \frac{1}{2}\right) = P\left(U > 2\theta k \mid \theta = \frac{1}{2}\right) = P(U > k)$$

$$K=79.08$$

let $\underline{X} = (X_1, X_2, \dots, X_n)$ random sample from $N(\theta, \sigma^2)$ where σ^2 known
Find γ_{GLR} and α for $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$

Maximum likelihood estimator of θ is $\bar{X}$

γ_{GLR} reject H_0 if $\lambda = \frac{\ell(\underline{X}; \theta_0)}{\ell(\underline{X}; \bar{X})} < k$

$$\lambda = \frac{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{1}{2} \frac{\sum (X_i - \theta_0)^2}{\sigma^2}\right)}{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{1}{2} \frac{\sum (X_i - \bar{X})^2}{\sigma^2}\right)}$$

$$\lambda = \exp\left(-\frac{1}{2} \frac{\sum (X_i - \theta_0)^2 - \sum (X_i - \bar{X})^2}{\sigma^2}\right)$$

$$\lambda = \exp\left(-\frac{1}{2} \frac{\sum (X_i - \theta_0 - \bar{X} + \bar{X})^2 - \sum (X_i - \bar{X})^2}{\sigma^2}\right)$$

$$\lambda = \exp\left(-\frac{1}{2} \frac{\sum (\bar{X} - \theta_0)^2 + \sum (X_i - \bar{X})^2 - \sum (X_i - \bar{X})^2}{\sigma^2}\right)$$

$$\lambda = \exp\left(-\frac{1}{2} \frac{n(\bar{X} - \theta_0)^2}{\sigma^2}\right)$$

i.e

$$\exp\left(-\frac{1}{2} \frac{n(\bar{X} - \theta_0)^2}{\sigma^2}\right) < k$$

$$-\frac{1}{2} \frac{n(\bar{X} - \theta_0)^2}{\sigma^2} < \ln k$$

$$\frac{n(\bar{X} - \theta_0)^2}{\sigma^2} > -2 \ln k = d^2$$

Then γ_{GLR} reject H_0 if

$$\begin{aligned} \alpha &= P(\lambda < k | \theta_0) \\ &= P\left(\frac{n(\bar{X} - \theta_0)^2}{\sigma^2} > d^2\right) \\ &= P\left(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > d\right) + P\left(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} < -d\right) \end{aligned}$$

When H_0 true then $\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \sim N(0,1)$ and $d = z_{1-\frac{\alpha}{2}}$

We accept H_0 if

$$\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \in \left(-z_{1-\frac{\alpha}{2}}, z_{1-\frac{\alpha}{2}}\right)$$

let $\underline{X} = (X_1, X_2, \dots, X_n)$ random sample from $N(\theta, \sigma^2)$ where σ^2 known
Find γ_{CI} and α for $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$

$$100(1 - \alpha) \text{ C.I. is } \left(T_1(\underline{X}), T_2(\underline{X})\right) = \left(\bar{X} - z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right)$$

Then we accept H_0 if

$$\theta_0 \in \left(\bar{X} - z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right) \Leftrightarrow \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \in \left(-z_{1-\frac{\alpha}{2}}, z_{1-\frac{\alpha}{2}}\right)$$