Chapter 5 :Testing Hypothesis

There are four possible situations that determines our decision is correct or in error. These four situations are summarized below:

	H_o is true	H_o is false
Accept H_o	Correct Decision	Type II Error
Reject H_o	Type I Error	Correct Decision

- The error type 1 of the testγ is: reject H_0 when it is true .
- the error type 2 of the test γ is :accept H_0 when it is false.
- The significance level (α) : P(Type I error)

  $$\alpha = P(Reject H_0|H_0 true)$$

  This is also equivalent to

  $$\alpha = P(Accept H_a|H_0 true)$$

- (β) : P(Type II error)

  $$\beta = P(Accept H_0|H_0 false)$$

  Similarly, this is also equivalent to

  $$\beta = P(Accept H_0|H_a true)$$

- The power function of a hypothesis test

  $$\pi(\theta) = \begin{cases} P(Reject H_0|H_a True). \\ 1 - P(Type II error) = 1 - \beta. \end{cases}$$

Example 1 : Let $X_1, X_2, ..., X_{20}$ be a random sample from a distribution with probability density function

$$f(x;p) = \begin{cases} p^x(1-p)^{1-x} & \text{if } x = 0, 1. \\ 0 & \text{otherwise.} \end{cases}$$

where $0 < p \le \frac{1}{2}$ is a parameter. The hypothesis $H_0 : p = \frac{1}{2}$ to be tested against $H_a : p < \frac{1}{2}$. If H_0 is rejected when $\sum_{i=1}^{20} X_i \le 6$, then what is the probability of type I error?

Solution 1:

Since each observation $X_i \sim BER(p)$,the sum the observations $\sum_{i=1}^{20} X_i \sim BIN(20, p)$. The probability of type I error is given by:

$$\begin{aligned} \alpha &= P(Type I Error) \\ &= P(Reject H_0|H_0 true) \\ &= P(\sum_{i=1}^{20} X_i \le 6|H_0 : p = \frac{1}{2}) \\ &= \sum_{i=0}^{6} \binom{20}{k} (\frac{1}{2})^k (1 - \frac{1}{2})^{20-k} \\ &= 0.0577 \end{aligned}$$

Hence the probability of type I error is 0.0577.

Example 2 : Suppose X has the density function

$$f(x) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < x < \theta. \\ 0 & \text{otherwise.} \end{cases}$$

If one observation of X is taken, what are the probabilities of Type I and Type II errors in testing the null hypothesis $H_0 : \theta = 1$ against the alternative hypothesis $H_a : \theta = 2$, if H_0 is rejected for $X > 0.92$.

Solution 2:
The probability of type I error is given by:

$$\begin{aligned} \alpha &= P(Type I Error) \\ &= P(Reject H_0 | H_0 true) \\ &= P(X > 0.92 | H_0 : \theta = 1) \\ &= \int_{0.92}^{1} 1 \, \mathrm{d}x \\ &= x \Big|_{0.92}^{1} \\ &= 0.08 \end{aligned}$$

The probability of type II error is given by:

$$\begin{aligned} \beta &= P(Type II Error) \\ &= P(Accept H_0 | H_0 False) \\ &= P(Accept H_0 | H_a True) \\ &= P(X \leq 0.92 | H_a : \theta = 2) \\ &= \int_{0}^{0.92} \frac{1}{2} \, \mathrm{d}x \\ &= \frac{x}{2} \Big|_{0}^{0.92} \\ &= 0.46 \end{aligned}$$

Hence the probability of type I error is 0.08 and the probability of type II error is 0.46.

Example 3 : Let $X_1, X_2, ..., X_8$be a random sample of size 8 from a Poisson distribution with parameter λ. Reject the null hypothesis $H_0 : \lambda = 0.5$ if the observed sum $\sum_{i=1}^{8} x_i \geq 8$. $H_a : \lambda \neq 0.5$.
First, compute the significance level α of the test.
Second,find the power function $\pi(\lambda)$ of the test as a sum of Poisson probabilities when H_a is true.
Solution 3:
significance level α :

$$\begin{aligned} \alpha &= P(Type I Error) \\ &= P(Reject H_0 | H_0 true) \\ &= P(\sum_{i=1}^{8} x_i \geq 8 | H_0 : \lambda = 0.5) \\ &= P(y \geq 8) \\ &= 1 - P(y < 8) \\ &= 1 - \sum_{y=0}^{7} \frac{4^y e^{-4}}{y!} \\ &= 0.0511 \end{aligned}$$

power function $\pi(\lambda)$ of the test:

$$\begin{aligned}
\pi(\lambda) &= P(RejectH_0|H_a true)\\
&= P(RejectH_0|H_a true)\\
&= P(\sum_{i=1}^{8} x_i \geq 8|H_a : \lambda \neq 0.5)\\
&= P(y \geq 8)\\
&= 1 - P(y < 8)\\
&= 1 - \sum_{y=0}^{7} \frac{(n\lambda)^y e^{-n\lambda}}{y!} \quad ; \quad (where \lambda \neq 0.5)
\end{aligned}$$

Example 4 : class activity

A normal population has a standard deviation of 16. The critical region for testing $H_0 : \mu = 5$ versus the alternative $H_a : \mu = k$ is $\bar{X} > k - 2$. What would be the value of the constant k and the sample size n which would allow the probability of Type I error to be 0.0228 and the probability of Type II error to be 0.1587.

Solution 4:

$X \sim N(\mu, 16^2)$

$$\begin{aligned}
\alpha &= P(TypeIError)\\
0.0228 &= P(\bar{x} > k - 2|\mu = 5)\\
0.0228 &= P(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} > \frac{k - 2 - 5}{16/\sqrt{n}})\\
0.0228 &= P(Z > \frac{k - 7}{16/\sqrt{n}})\\
0.0228 &= 1 - P(Z < \frac{k - 7}{16/\sqrt{n}})\\
P(Z < \frac{k - 7}{16/\sqrt{n}}) &= 0.9772
\end{aligned}$$

Hence ,from standard normal table , we have :

$$\frac{k - 7}{16/\sqrt{n}} = 2$$

which gives

$$(k - 7)\sqrt{n} = 32$$

Similarly,

$$\begin{aligned}
\beta &= P(TypeIIError)\\
\beta &= P(AcceptH_0|H_0 False)\\
\beta &= P(AcceptH_0|H_a True)\\
0.1587 &= P(\bar{x} > k - 2|\mu = k)\\
0.1587 &= P(Z < \frac{k - 2 - k}{16/\sqrt{n}})\\
0.1587 &= P(Z < \frac{-2}{16/\sqrt{n}})
\end{aligned}$$

Hence ,from standard normal table , we have :

$$\begin{aligned} \frac{-2}{16/\sqrt{n}} &= -1 \\ 2\sqrt{n} &= 16 \\ \sqrt{n} &= 8 \\ n &= 8^2 = 64 \end{aligned}$$

Letting this value of n in :

$$\begin{aligned} (k-7)\sqrt{n} &= 32 \\ (k-7)\sqrt{64} &= 32 \\ k &= \frac{32}{8} + 7 \end{aligned}$$

We see that $k = 11$.

Example 5 : Homework
A random sample of size 4 is taken from a normal distribution with unknown mean μ and variance $\sigma^2 > 0$. To test $H_0 : \mu = 0$ against $H_a : \mu < 0$ the following test is used: Reject H_0 if and only if $X_1 + X_2 + X_3 + X_4 < -20$. Find the value of σ so that the significance level of this test will be closed to 0.14.
Solution 5:
Sice $\alpha = 0.14$

$$\begin{aligned} \alpha &= P(TypeIError) \\ 0.14 &= P(RejectH_0|H_0true) \\ 0.14 &= P(X_1 + X_2 + X_3 + X_4 < -20|H_0 : \mu = 0) \\ 0.14 &= P(\overline{X} < \frac{-20}{4}|H_0 : \mu = 0) \\ 0.14 &= P(Z < \frac{-5-0}{\sigma/\sqrt{4}}) \end{aligned}$$

we get from the standard normal table :

$$\begin{aligned} \frac{-5-0}{\sigma/2} &= -1.08 \\ \frac{-10}{\sigma} &= -1.08 \\ \sigma &= 9.259 \end{aligned}$$

Definition

A distribution $f(x;\theta)$ belongs to the class of exponential families if, it is written in the form:

$$f(x;\theta) = e^{a(\theta)+b(x)+d(x)c(\theta)}.$$

Example 6 :
Show that the Exponential distribution belong to the exponential family:

$$f(y;\theta) = \begin{cases} \theta e^{-y\theta} & \text{if } y > 0, \theta > 0. \\ 0 & \text{otherwise.} \end{cases}$$

Solution 6:

$$\begin{aligned} f(y;\theta) &= \theta e^{-y\theta} \\ &= e^{\log(\theta)-y\theta} \\ &= e^{a(\theta)+b(y)+d(y)c(\theta)} \end{aligned}$$

where

$$\begin{aligned} a(\theta) &= \log(\theta) \\ b(y) &= 0 \\ c(\theta) &= -\theta \\ d(y) &= y \end{aligned}$$

Theorem

If $f(x;\theta)$ belongs to the class of exponential families, then the test γ_{MP} for $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ rejects H_0 is reduced as follows:

	$\theta_0 < \theta_1$	$\theta_0 > \theta_1$
$c(\theta) \nearrow$	$\sum d(x_i) > k$	$\sum d(x_i) < k$
$c(\theta) \searrow$	$\sum d(x_i) < k$	$\sum d(x_i) > k$
	Alike	Inverse

k solves the equation

$$\alpha_{MP} = \mathbf{P}\left(Reject \quad H_0|\theta_0\right).$$

Example 7 :
Let X be normal random variable with distribution $N(\theta, 1)$. Let $X_1, X_2, ..., X_{16}$ be 16 copies of X. Test the hypothesis $H_0 : \theta = 0$ vs $H_a : \theta = 1$ by γ_{MP} with size $\alpha_{MP} = 0.05$.
Solution 7:

$$f(x,\theta) = \frac{1}{\sqrt{2\Pi}} e^{\frac{1}{2}(x-\theta)^2}; -\infty < x < \infty$$

$f(x;\theta)$ belong to the class of exponential families :

$$\begin{aligned} f(x;\theta) &= e^{-\frac{1}{2}\log(2\Pi)-\frac{1}{2}(x-\theta)^2} \\ &= e^{-\frac{1}{2}\log(2\Pi)-\frac{1}{2}(x^2-2x\theta+\theta^2)} \\ &= e^{-\frac{1}{2}\log(2\Pi)-\frac{1}{2}x^2+x\theta-\frac{\theta^2}{2}} \end{aligned}$$

Hence

$$\begin{aligned} a(\theta) &= -\frac{\theta^2}{2} \\ b(x) &= -\frac{1}{2}\log(2\Pi) - \frac{1}{2}x^2 \\ c(\theta) &= \theta \\ d(x) &= x \end{aligned}$$

Since $c(\theta)$ is an increasing function , then γ_{MP} reject H_0 if $\sum d(x) > k$:

$$\Rightarrow \quad Reject \quad H_0 \quad if \sum x > k$$

To find the value of k :

$$\begin{aligned} \alpha_{MP} &= P(TypeIError) \\ 0.05 &= P(\sum x > k|\theta = 0) \\ 0.05 &= P(\overline{x} > \frac{k}{16}|\theta = 0) \\ 0.05 &= P(Z > \frac{\frac{k}{16} - 0}{1/\sqrt{16}}) \\ 0.05 &= P(Z > \frac{\frac{k}{16}}{1/4}) \\ 0.05 &= P(Z > \frac{4k}{16}) \\ 0.05 &= 1 - P(Z < \frac{4k}{16}) \\ P(Z < \frac{4k}{16}) &= 0.95 \end{aligned}$$

from the standard normal table :

$$\begin{aligned} \frac{4k}{16} &= 1.645 \\ k &= 6.58 \end{aligned}$$

7.a : If $\sum_{i=1}^{16} x_i = 10$, what is your conclusion ? .

A. *Accept* H_0 **B.** <u>*Reject* H_0</u>

Reject H_0 if $\sum x > 6.58 \Rightarrow \sum x = 10 > 6.58 \Rightarrow$ "Reject H_0 .

7.b : Compute the probability of Type II error ? .

$$\begin{aligned} \beta_{MP} &= P(TypeIIError) \\ \beta_{MP} &= P(AcceptH_0|H_aTrue) \\ &= P(\sum x \leq 6.58|\theta = 1) \\ &= P(\bar{x} \leq \frac{6.58}{16}|\theta = 1) \\ &= P(\bar{x} \leq 0.41125|\theta = 1) \\ &= P(Z \leq \frac{0.41125 - 1}{1/\sqrt{16}}) \\ &= P(Z \leq -2.355) \end{aligned}$$

from the standard normal table :

$$\beta_{MP} = 0.00914$$

Example 8 :
Let X be gamma random variable with distribution $Gamma(5, \theta)$. Let $X_1, X_2, ..., X_6$ be 6 copies of X. Test the hypothesis$H_0 : \theta = 1$ vs $H_a : \theta = \frac{1}{2}$ by γ_{MP} with size $\alpha_{MP} = 0.05$.
Solution 8:

$$f(x, \theta) = \frac{\theta^5}{\Gamma(5)} x^{5-1} e^{-\theta x}$$

$f(x; \theta)$ belong to the class of exponential families :

$$f(x; \theta) = e^{5\log(\theta) + 4\log(x) - \theta x - \log(\Gamma 5)}$$

Hence

$$\begin{aligned} a(\theta) &= 5\log(\theta) \\ b(x) &= 4\log(x) - \log(\Gamma 5) \\ c(\theta) &= -\theta \\ d(x) &= x \end{aligned}$$

Since $c(\theta)$ is a decreasing function , then γ_{MP} reject H_0 if $\sum d(x) > k$:

$$\Rightarrow \quad Reject \quad H_0 \quad if \sum x > k$$

To find the value of k :

$$\begin{aligned} \alpha_{MP} &= P(Type I Error) \\ 0.05 &= P(\sum x > k|\theta = 1), \quad let \quad y = \sum x \quad where \quad x \sim Gamma(5, \theta) \\ 0.05 &= P(y > k); \quad y \sim Gamma(n5, \theta) \Rightarrow \quad y \sim Gamma(30, 1) \end{aligned}$$

note: If $Y \sim Gamma(n, \theta), then \quad T(X) = 2\theta Y \sim \chi^2_{2n}$.
then

$$\begin{aligned} y \sim Gamma(n = 30, \theta = 1) \Rightarrow U &= 2\theta y \sim \chi^2_{2n} \\ U &= 2(1)y \sim \chi^2_{2(30)} \end{aligned}$$

$$\begin{aligned} 0.05 &= P(y > k) \\ 0.05 &= P(U > 2k); \quad U \sim \chi^2_{60} \end{aligned}$$

From Chi-square table :

$$\begin{aligned} 2k &= 79.08 \\ k &= 39.54 \end{aligned}$$

8.a : Compute the probability of Type II error ? Homework .

$$
\begin{aligned}
\beta_{MP} &= P(TypeIIError)\\
\beta_{MP} &= P(AcceptH_0|H_aTrue)\\
&= P(y < 39.54|\theta = \frac{1}{2})\\
&= P(U < 39.54); \quad U \sim \chi^2_{60}\\
&= 1 - P(U > 39.54)\\
&= 1 - (\frac{0.99 + 0.98}{2}), \quad From \quad Chi - square \quad table\\
\Rightarrow \beta_{MP} &= 0.015
\end{aligned}
$$

Degree of Freedom	Probability of Exceeding the Critical Value								
	0.99	0.95	0.90	0.75	0.50	0.25	0.10	0.05	0.01
1	0.000	0.004	0.016	0.102	0.455	1.32	2.71	3.84	6.63
2	0.020	0.103	0.211	0.575	1.386	2.77	4.61	5.99	9.21
3	0.115	0.352	0.584	1.212	2.366	4.11	6.25	7.81	11.34
4	0.297	0.711	1.064	1.923	3.357	5.39	7.78	9.49	13.28
5	0.554	1.145	1.610	2.675	4.351	6.63	9.24	11.07	15.09
6	0.872	1.635	2.204	3.455	5.348	7.84	10.64	12.59	16.81
7	1.239	2.167	2.833	4.255	6.346	9.04	12.02	14.07	18.48
8	1.647	2.733	3.490	5.071	7.344	10.22	13.36	15.51	20.09
9	2.088	3.325	4.168	5.899	8.343	11.39	14.68	16.92	21.67
10	2.558	3.940	4.865	6.737	9.342	12.55	15.99	18.31	23.21
11	3.053	4.575	5.578	7.584	10.341	13.70	17.28	19.68	24.72
12	3.571	5.226	6.304	8.438	11.340	14.85	18.55	21.03	26.22
13	4.107	5.892	7.042	9.299	12.340	15.98	19.81	22.36	27.69
14	4.660	6.571	7.790	10.165	13.339	17.12	21.06	23.68	29.14
15	5.229	7.261	8.547	11.037	14.339	18.25	22.31	25.00	30.58
16	5.812	7.962	9.312	11.912	15.338	19.37	23.54	26.30	32.00
17	6.408	8.672	10.085	12.792	16.338	20.49	24.77	27.59	33.41
18	7.015	9.390	10.865	13.675	17.338	21.60	25.99	28.87	34.80
19	7.633	10.117	11.651	14.562	18.338	22.72	27.20	30.14	36.19
20	8.260	10.851	12.443	15.452	19.337	23.83	28.41	31.41	37.57
22	9.542	12.338	14.041	17.240	21.337	26.04	30.81	33.92	40.29
24	10.856	13.848	15.659	19.037	23.337	28.24	33.20	36.42	42.98
26	12.198	15.379	17.292	20.843	25.336	30.43	35.56	38.89	45.64
28	13.565	16.928	18.939	22.657	27.336	32.62	37.92	41.34	48.28
30	14.953	18.493	20.599	24.478	29.336	34.80	40.26	43.77	50.89
40	22.164	26.509	29.051	33.660	39.335	45.62	51.80	55.76	63.69
50	27.707	34.764	37.689	42.942	49.335	56.33	63.17	67.50	76.15
60	37.485	43.188	46.459	52.294	59.335	66.98	74.40	79.08	88.38

39.54

Neyman-Pearson lemma

The test γ_{MP} of size α_{MP} is found by the following steps:

1. Take the Likelihood Ratio (LR) $\lambda = \frac{\ell(\underline{X};\theta_0)}{\ell(\underline{X};\theta_1)}$.
2. Reject $H_0 : \theta = \theta_0$ if $\lambda < k$.
3. Find k by solving the implicit equation $\alpha_{MP} = \mathbf{P}(\lambda < k|\theta_0)$.

Example 9:

Suppose X has the density function

$$
f(y;\theta) = \begin{cases} (1+\theta)x^{\theta} & \text{if } 0 \le x \le 1. \\ 0 & \text{otherwise.} \end{cases}
$$

Based on a single observed value of X, find the most powerful critical region of size $\alpha = 0.1$ for testing $H_0 : \theta = 1$ against $H_a : \theta = 2$.(Use Neyman -Pearson lemma)

Solution 9:

By Neyman-Pearson Theorem, the form of the critical region is given by:

1- Take likelihood ratio (LR) :

$$\begin{aligned}\lambda &= \frac{\ell(\underline{X}, \theta_0)}{\ell(\underline{X}, \theta_1)} \\ \lambda &= \frac{(1+\theta_0)x^{\theta_0}}{(1+\theta_1)x^{\theta_1}} \\ &= \frac{2x}{3x^2} \\ &= \frac{2}{3x}\end{aligned}$$

2- Reject $H_0 : \theta = \theta_0$ if $\lambda < K$:

$$\begin{aligned}\mathcal{C} &= \{\frac{2}{3x} < K\} \\ &= \{\frac{1}{x} < \frac{3}{2}K\} \\ &= \{x > a\}\end{aligned}$$

where a is some constant. Hence the most powerful or best test is of the form: **Reject Ho if** $X > a$. Since, the significance level of the test is given to be $\alpha = 0.1$, the constant a can be determined. Now we proceed to find a. Since

$$\begin{aligned}\alpha &= P(TypeIError) \\ 0.1 &= P(RejectH_0|H_0true) \\ &= P(X > a|H_0 : \theta = 1) \\ &= \int_a^1 2x \, \mathrm{d}x \\ &= 1 - a^2\end{aligned}$$

hence

$$a^2 = 1 - 0.1 = 0.9$$

Therefor

$$a = \sqrt{0.9}$$