

Chapter 4 : Approximation of the confidence interval

In this section, we discuss how to construct an approximate $(1 - \alpha)100\%$ confidence interval for a population parameter θ using its maximum likelihood estimator $\hat{\theta}$.

The confidence interval is obtained by the following steps:

(1) $\ell(X, \theta) = \prod_{i=1}^n f(x_i, \theta)$.

(2) $L(X; \theta) = \log(\ell(X, \theta))$.

(3) if $\frac{\partial^2 L(X; \theta)}{\partial \theta^2} < 0$, then $\hat{\theta}_{MLE}$ is the solution of this equation:

$$\frac{\partial L(X; \theta)}{\partial \theta} = 0$$

(4) The Fisher information is $I_n = \mathbf{E}(-\frac{\partial^2 L(X; \theta)}{\partial \theta^2})$.

Then the confidence interval of θ is given by:

$$(\hat{\theta}_{MLE} \pm \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{\hat{I}_n}})$$

Example 1 : Let X be Gamma random variable with distribution :

$$f(x, \theta) = \frac{\theta^r}{\Gamma(r)} x^{r-1} e^{-\theta x}; x \geq 0$$

Let $X = (X_1, \dots, X_n)$ be n copies of X . Find an approximation C.I $(1 - \alpha)100\%$ of θ .

Solution 1 :

First, find the maximum likelihood estimator of θ :

$$\begin{aligned} \ell(X, \theta) &= \prod_{i=1}^n f(x_i, \theta) \\ &= \left(\frac{\theta^r}{\Gamma(r)}\right)^n \left(\prod_{i=1}^n x_i\right)^{r-1} e^{-\theta \sum_{i=1}^n x_i} \end{aligned}$$

Then

$$\begin{aligned} L(X; \theta) &= \log(\ell(X, \theta)) \\ &= nr \log(\theta) - n \log(\Gamma(r)) + (r-1) \sum_{i=1}^n \log(x_i) - \theta \sum_{i=1}^n x_i \end{aligned}$$

The first derivative of the logarithm of likelihood function is:

$$\frac{\partial L(X; \theta)}{\partial \theta} = \frac{nr}{\theta} - \sum_{i=1}^n x_i$$

Setting this derivative to zero and solving for θ :

$$\frac{\partial L(X; \theta)}{\partial \theta} = 0$$

Hence, the maximum likelihood estimator of θ is given by:

$$\hat{\theta} = \frac{r}{\bar{x}}$$

The second derivative of the logarithm of the likelihood function is given by:

$$\frac{\partial^2 L(X; \theta)}{\partial \theta^2} = \frac{-nr}{\theta^2}$$

The fisher information of θ is given by:

$$\begin{aligned} I_n &= \mathbf{E}\left(-\frac{\partial^2 L(X; \theta)}{\partial \theta^2}\right) \\ &= \mathbf{E}\left(\frac{nr}{\theta^2}\right) \\ &= \frac{nr}{\theta^2} \end{aligned}$$

we replace the unknown θ by its estimate $\hat{\theta}$. ($\hat{\theta} = \frac{r}{\bar{x}}$)

$$\hat{I}_n = \frac{n\bar{x}^2}{r}$$

The $100(1 - \alpha)$ % approximate confidence interval for θ is:

$$\begin{aligned} (\hat{\theta}_{MLE} \pm \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{\hat{I}_n}}) \\ (\frac{r}{\bar{x}} \pm \frac{Z_{1-\frac{\alpha}{2}}}{\bar{x}\sqrt{\frac{n}{r}}}) \end{aligned}$$

Example 2 : Let X be geometric random variable with distribution :

$$f(x, \theta) = \theta (1 - \theta)^{x-1}; x = 1, 2, 3, \dots$$

let $X = (X_1, \dots, X_n)$ be n copies of X . Find an approximation $100(1 - \alpha)$ % of θ .

Solution 2 :

First ,find the maximum likelihood estimator of θ :

$$\begin{aligned} \ell(X, \theta) &= \prod_{i=1}^n f(x_i, \theta) \\ &= (\theta)^n (1 - \theta)^{\sum_{i=1}^n x_i - n} \end{aligned}$$

Then

$$\begin{aligned} L(X; \theta) &= \log(\ell(X, \theta)) \\ &= n \log(\theta) + \left(\sum_{i=1}^n x_i - n\right) \log(1 - \theta) \end{aligned}$$

The first derivative of the logarithm of likelihood function is:

$$\frac{\partial L(X; \theta)}{\partial \theta} = \frac{n}{\theta} - \frac{\sum_{i=1}^n x_i - n}{1 - \theta}$$

Setting this derivative to zero and solving for θ :

$$\frac{\partial L(X; \theta)}{\partial \theta} = 0$$

Then

$$\hat{\theta} = \frac{1}{\bar{x}}$$

we need the second derivative of $L(X; \theta)$:

$$\begin{aligned} \frac{\partial^2 L(X; \theta)}{\partial \theta^2} &= \frac{-n}{\theta^2} - \frac{\sum_{i=1}^n x_i - n}{(1 - \theta)^2} \\ &= -\left(\frac{n}{\theta^2} + \frac{\sum_{i=1}^n x_i - n}{(1 - \theta)^2}\right) \end{aligned}$$

The fisher information of θ is given by:

$$\begin{aligned} I_n &= \mathbf{E}\left(-\frac{\partial^2 L(X; \theta)}{\partial \theta^2}\right) \\ &= \mathbf{E}\left(\frac{n}{\theta^2} + \frac{\sum_{i=1}^n x_i - n}{(1-\theta)^2}\right) \end{aligned}$$

mean of a geometric distribution is given as follows: $\mathbf{E}(x) = \frac{1}{\theta}$ <https://www.semaph.info/src/st-geometric-distribution.html>

$$\begin{aligned} I_n &= n\left[\frac{1}{\theta^2} - \frac{1}{(1-\theta)^2} + \frac{1}{\theta(1-\theta)^2}\right] \\ &= n\left[\frac{(1-\theta)^2 - \theta^2 + \theta}{\theta^2(1-\theta)^2}\right] \\ &= n\left[\frac{1-\theta}{\theta^2(1-\theta)^2}\right] \\ &= \frac{n}{\theta^2(1-\theta)} \end{aligned}$$

we replace the unknown θ by its estimate $\hat{\theta}$

$$\hat{I}_n = \frac{n}{\hat{\theta}^2(1-\hat{\theta})}$$

Thus, $100(1-\alpha)\%$ approximate confidence interval for θ is:

$$\begin{aligned} (\hat{\theta}_{MLE} \pm \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{\hat{I}_n}}) \\ (\hat{\theta} \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\theta}^2(1-\hat{\theta})}{n}}) \end{aligned}$$

Example 3 : If $X_1, X_2, \dots, X_n$ is a random sample from a population with density:

$$f(x; \theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1. \\ 0 & \text{otherwise.} \end{cases}$$

where $\theta > 0$ is an unknown parameter.

(a) what is a $100(1-\alpha)\%$ approximate confidence interval for θ if the sample size is large?

(b) Find 90% approximate C.I of θ , if $\sum_{i=1}^{49} \log x_i = -0.7564$

Solution 3:

(a) First, find the maximum likelihood estimator of θ :

$$\begin{aligned} \ell(X, \theta) &= \prod_{i=1}^n f(x_i, \theta) \\ &= (\theta)^n \prod_{i=1}^n x_i^{(\theta-1)} \end{aligned}$$

Then

$$\begin{aligned} L(X; \theta) &= \log(\ell(X, \theta)) \\ &= n \log(\theta) + (\theta - 1) \sum_{i=1}^n \log(x_i) \end{aligned}$$

The first derivative of the logarithm of likelihood function is:

$$\frac{\partial L(X; \theta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^n \log(x_i)$$

Setting this derivative to zero and solving for θ :

$$\frac{\partial L(X; \theta)}{\partial \theta} = 0$$

Hence, the maximum likelihood estimator of θ is given by:

$$\hat{\theta} = \frac{-n}{\sum_{i=1}^n \log(x_i)}$$

The second derivative of the logarithm of the likelihood function is given by:

$$\frac{\partial^2 L(X; \theta)}{\partial \theta^2} = \frac{-n}{\theta^2}$$

The fisher information of θ is given by:

$$\begin{aligned} I_n &= \mathbf{E}\left(-\frac{\partial^2 L(X; \theta)}{\partial \theta^2}\right) \\ &= \mathbf{E}\left(\frac{n}{\theta^2}\right) \\ &= \frac{n}{\theta^2} \end{aligned}$$

we replace the unknown θ by its estimate $\hat{\theta}$. ($\hat{\theta} = \frac{-n}{\sum \log x_i}$)

$$\hat{I}_n = \frac{(\sum_{i=1}^n \log x_i)^2}{n}$$

The $100(1 - \alpha)$ % approximate confidence interval for θ is:

$$\begin{aligned} &(\hat{\theta}_{MLE} \pm \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{\hat{I}_n}}) \\ &(\frac{-n}{\sum \log x_i} \pm \frac{Z_{1-\frac{\alpha}{2}} \sqrt{n}}{\sum \log x_i}) \end{aligned}$$

(b) We are given the followings:

$$\begin{aligned} n &= 49 \\ \sum_{i=1}^{49} \log x_i &= -0.7576 \\ 1 - \alpha &= 0.90 \end{aligned}$$

Hence, we get

$$\begin{aligned} Z_{0.95} &= 1.64 \\ \frac{n}{\sum \log x_i} &= \frac{49}{-0.7567} = -64.75 \end{aligned}$$

and

$$\frac{\sqrt{n}}{\sum \log x_i} = \frac{7}{-0.7567} = -9.25$$

Hence, the approximate confidence interval is given by

$$(64.75 - (1.64)(9.25), 64.75 + (1.64)(9.25))$$

that is (49.58, 79.92) .

Example 4 : class activity

If $X_1, X_2, \dots, X_n$ is a random sample from a population with density:

$$f(x; \theta) = \begin{cases} (1 - \theta)\theta^x & \text{if } x = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

where $0 < \theta < 1$ is an unknown parameter, what is a $100(1 - \alpha) \%$ approximate confidence interval for θ if the sample size is large?

Solution 4:

Thus $100(1 - \alpha) \%$ approximate confidence interval for θ is: https://proofwiki.org/wiki/Expectation_of_Geometric_Distribution/Proof_1

$$\begin{aligned} & (\hat{\theta}_{MLE} \pm \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{\hat{I}_n}}) \\ & (\hat{\theta} \pm Z_{1-\frac{\alpha}{2}} \frac{\hat{\theta}(1-\hat{\theta})}{\sqrt{n(1-\hat{\theta}+\hat{\theta}^2)}}) \end{aligned}$$

where

$$\hat{\theta} = \frac{\bar{x}}{1 + \bar{x}}$$

Example 5 : Homework

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a distribution with a probability density function:

$$f(x; \theta) = \begin{cases} (\theta + 1)x^{-\theta-2} & \text{if } 1 < x < \infty. \\ 0 & \text{otherwise.} \end{cases}$$

where $0 < \theta$ is a parameter, what is a $100(1 - \alpha) \%$ approximate confidence interval for θ if the sample size is large?

Solution 5:

$$\hat{\theta} \pm Z_{1-\frac{\alpha}{2}} \frac{\hat{\theta} + 1}{\sqrt{n}}$$

where $\hat{\theta} = -1 + \frac{n}{\sum_{i=1}^n \log x_i}$